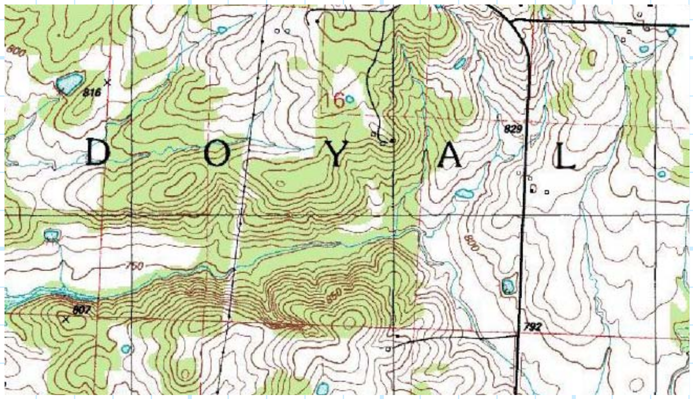
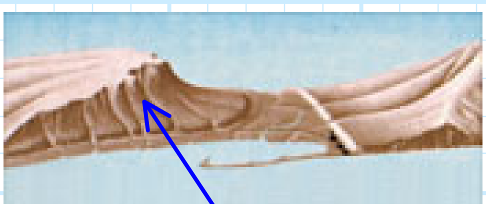
The Gradient

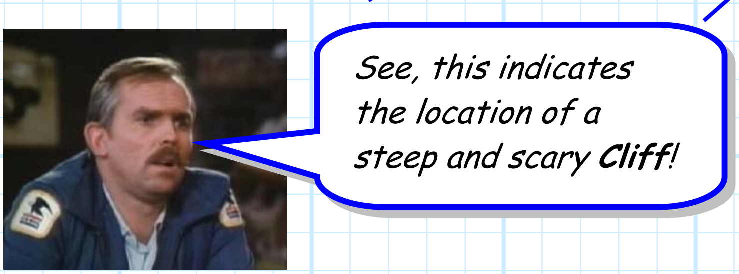
Consider the **topography** of the Earth's surface.



We use contours of constant elevation—called **topographic contours**—to express on maps (a 2-dimensional graphic) the third dimension of elevation (i.e., surface height).

We can infer from these maps the **slope** of the Earth's surface, as topographic contours lie closer together where the surface is very steep.





From: *erg.usgs.gov/isb/pubs/booklets/symbols/reading.html*

Moreover, we can likewise infer the **direction** of these slopes—a hillside might slope toward the south, or a cliff might drop-off toward the East.

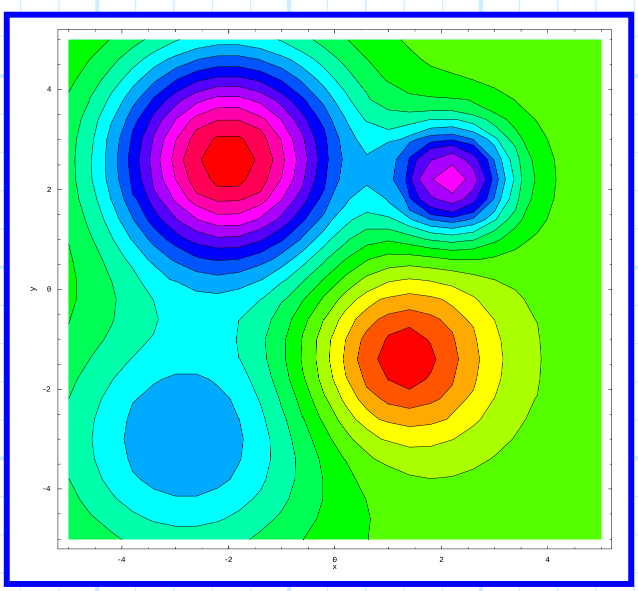
Thus, the slope of the Earth's surface has both a magnitude (e.g., flat or steep) and a direction (e.g. toward the north). In other words, the slope of the Earth's surface is a **vector quantity**!

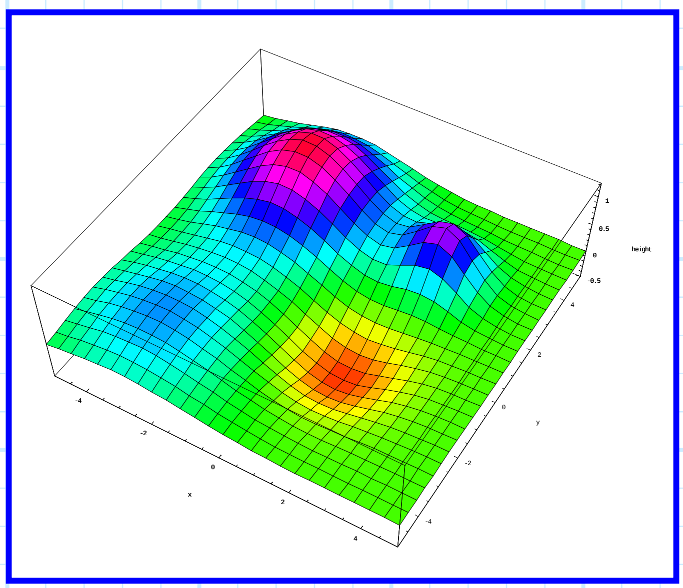
Thus, the surface slope at every point across some section of the Earth (e.g., Douglas County, Colorado, or North America) must be described by a **vector field**!

Q: *Sure, but there isn't any way to calculate this vector field is there?*

A: Yes, there is a very easy way, called the **gradient**.

Say the topography of some small section of the Earth's surface can be described as a **scalar** function $h(x,y)$, where h represents the **height** (elevation) of the Earth at some point denoted by coordinates x and y. E.G.:





Now say we take the **gradient** of scalar field *h*(*x*,*y*). We denote this operation as:

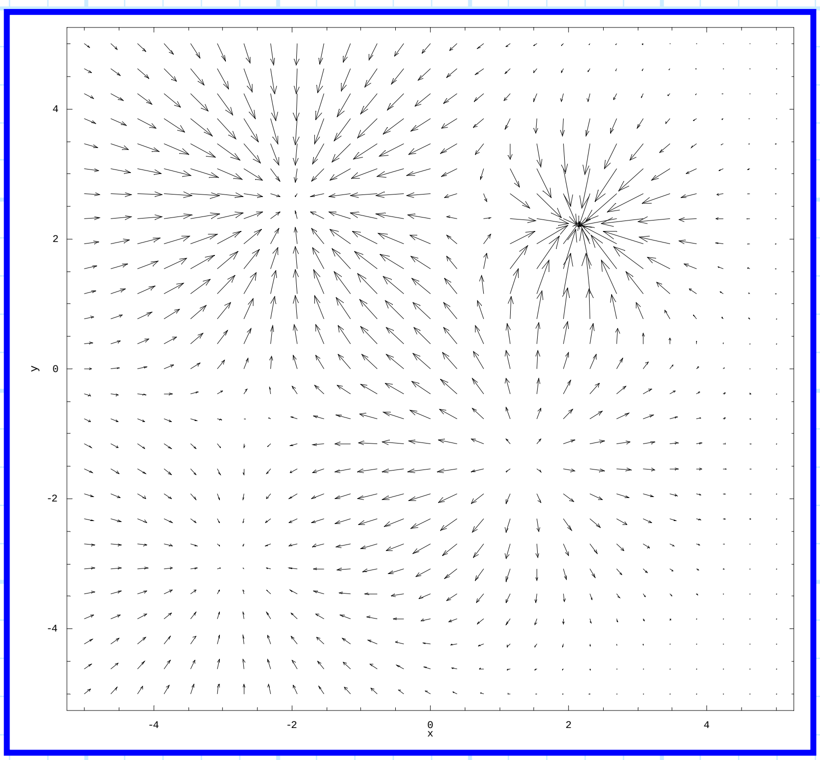
$$\nabla h(\bar{r})$$

The result of taking the gradient of a scalar field is a **vector field**, i.e.:

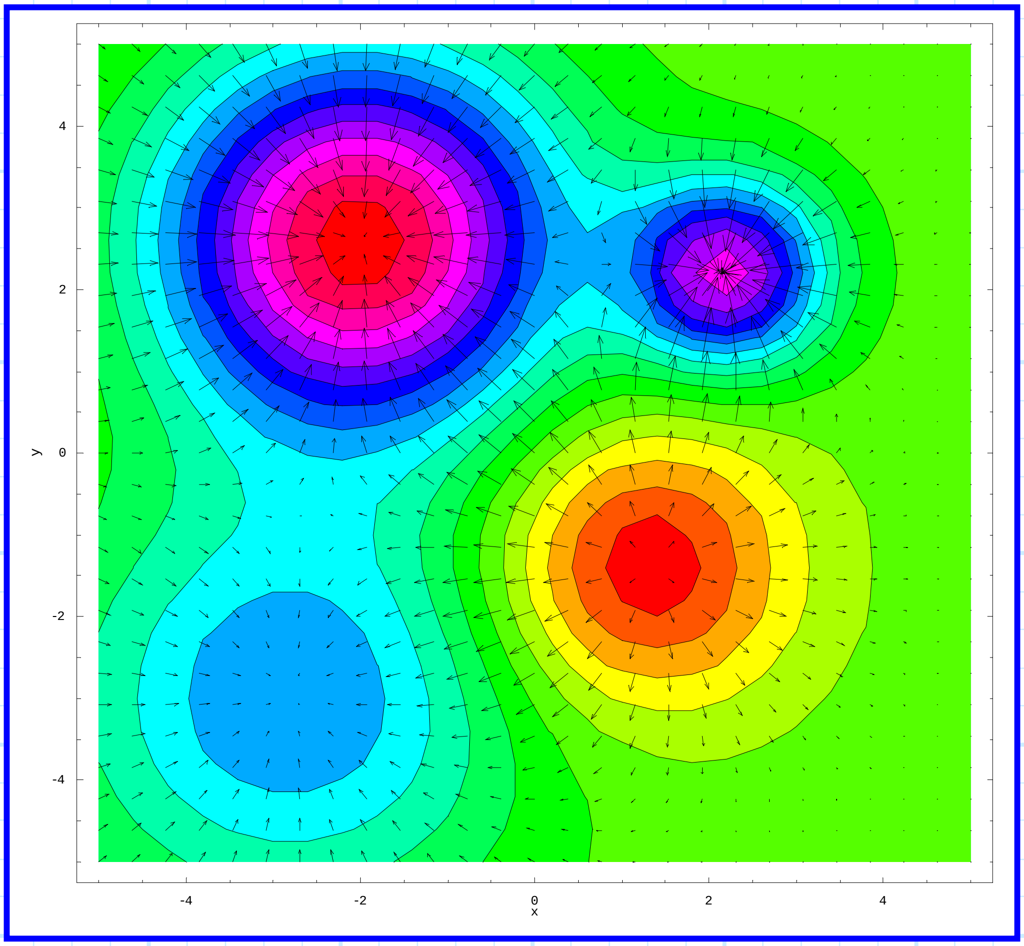
$$\nabla h(\bar{r}) = \mathbf{A}(\bar{r})$$

Q: *So just what **is** this resulting vector field, and how does it **relate** to scalar field $h(\bar{r})$??*

For our example here, taking the **gradient** of surface elevation *h*(*x*,*y*) results in the following **vector** field:



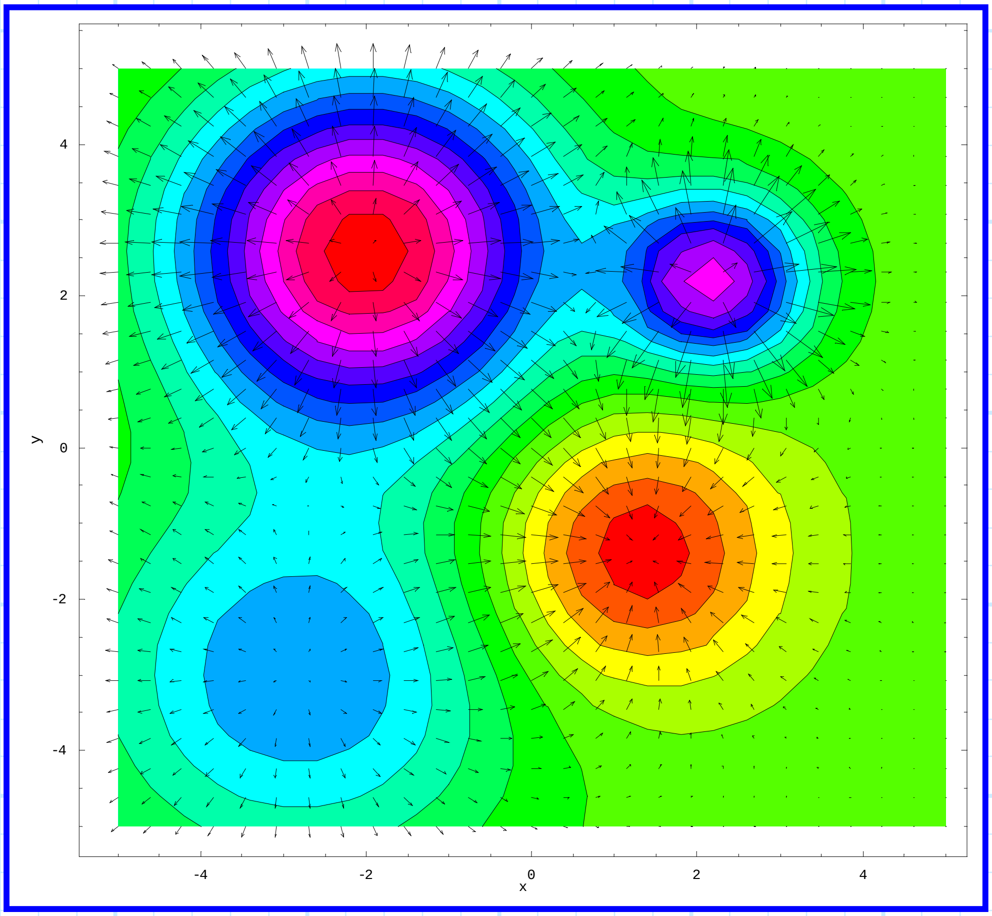
To see how this **vector** field **relates** to the surface height $h(x,y)$, let's place the vector field on top of the topographic plot:



Q: *It appears that the vector field indicates the slope of the surface topology—both its **magnitude** and **direction**!*

A: That's right! The gradient of a **scalar** field provides a **vector** field that states how the scalar value is **changing** throughout space—a change that has both a **magnitude** and **direction**.

It is a bit more "natural" and instructive for our example to examine the **opposite** of the gradient of h(x,y) (i.e., $\mathbf{A}(\bar{r}) = -\nabla h(\bar{r})$). In other words, to plot the vectors such that they are pointing in the "**downhill**" direction.



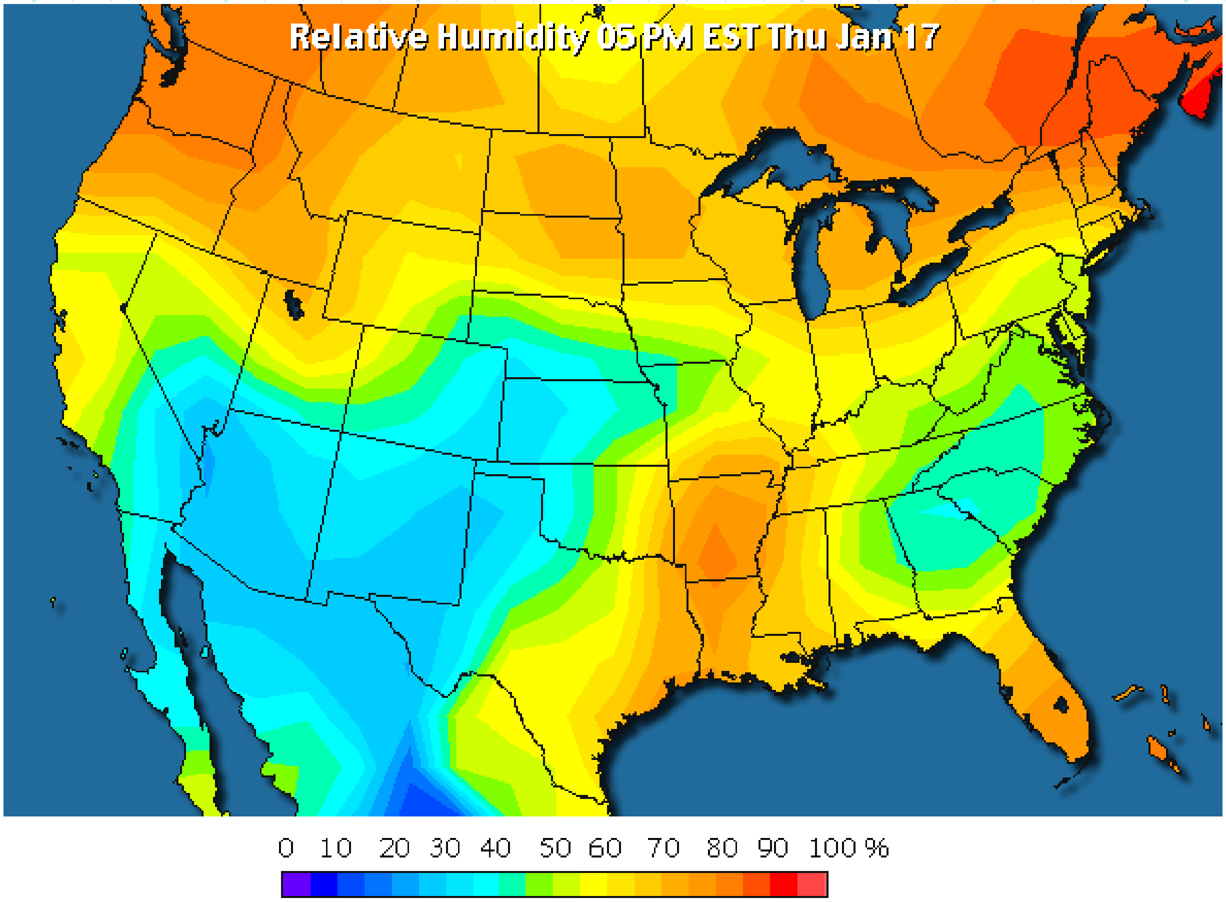
Note these important facts:

* The vectors point in the direction of **maximum change** (i.e., they point straight down the mountain!).

* The vectors always point **orthogonal** to the topographic contours (i.e., the contours of equal surface height).

Now, it is important to understand that the scalar fields we will consider will **not** typically describe the height or altitude of anything! Thus, the slope provided by the gradient is more mathematically "abstract", in the same way we speak about the slope (i.e., derivative) of some curve.

For example, consider the **relative humidity** across the country—a **scalar** function of position.



If we travel in some directions, we will find that the humidity quickly changes. But if we travel in other directions, the humidity will change not at all.

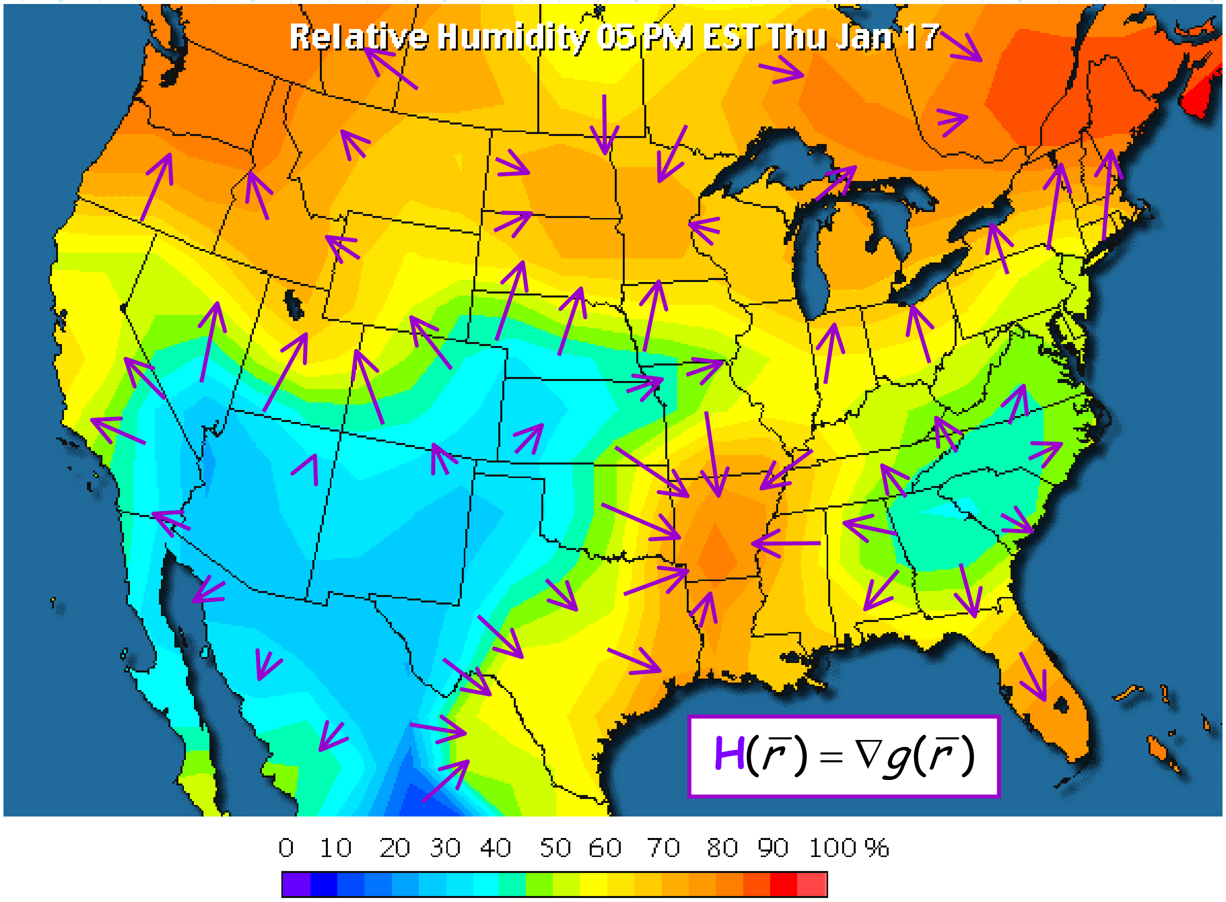
Q: *Say we are located at some point (e.g., Lawrence, KS; Albuquerque, N M; or Ann Arbor, MI), how can we determine the direction where we will experience the greatest change in humidity ?? Also, how can we determine what that change will be ??*

A: The answer to both questions is to take the **gradient** of the scalar field that represents humidity!

If $g(\bar{\mathrm{r}})$ is the scalar field that represents the humidity across the country, then we can form a vector field $\mathbf{H}(\bar{\mathrm{r}})$ by taking the gradient of $g(\bar{\mathrm{r}})$:

$$\mathbf{H}(\bar{\mathrm{r}}) = \nabla g(\bar{\mathrm{r}})$$

This vector field indicates the **direction** of greatest humidity change (i.e., the direction where the derivative is the largest), as well as the **magnitude** of that change, at every point in the country!



This is likewise true for **any** scalar field. The gradient of a scalar field produces a **vector** field indicating the direction of greatest change (i.e., largest derivative) as well as the magnitude of that change, at every point in space.

The Gradient Operator in Coordinate Systems

For the **Cartesian** coordinate system, the Gradient of a scalar field is expressed as:

$$\nabla g(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x}\,\hat{a}_x + \frac{\partial g(\bar{r})}{\partial y}\,\hat{a}_y + \frac{\partial g(\bar{r})}{\partial z}\,\hat{a}_z$$

Now let's consider the gradient operator in the **other** coordinate systems.

Q: *Pfft! This is easy! The gradient operator in the spherical coordinate system is:*

$$\nabla g(\bar{\text{r}}) = \frac{\partial g(\bar{\text{r}})}{\partial r}\,\hat{a}_r + \frac{\partial g(\bar{\text{r}})}{\partial \theta}\,\hat{a}_\theta + \frac{\partial g(\bar{\text{r}})}{\partial \phi}\,\hat{a}_\phi$$

Right ??

A: **NO!!** The above equation is **not** correct!

Instead, we find that for **spherical** coordinates, the gradient is expressed as:

$$\nabla g(\bar{r}) = \frac{\partial g(\bar{r})}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial g(\bar{r})}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial g(\bar{r})}{\partial \phi}\hat{a}_\phi$$

And for the **cylindrical** coordinate system we likewise get:

$$\nabla g(\bar{r}) = \frac{\partial g(\bar{r})}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial g(\bar{r})}{\partial \phi}\hat{a}_\phi + \frac{\partial g(\bar{r})}{\partial z}\hat{a}_z$$

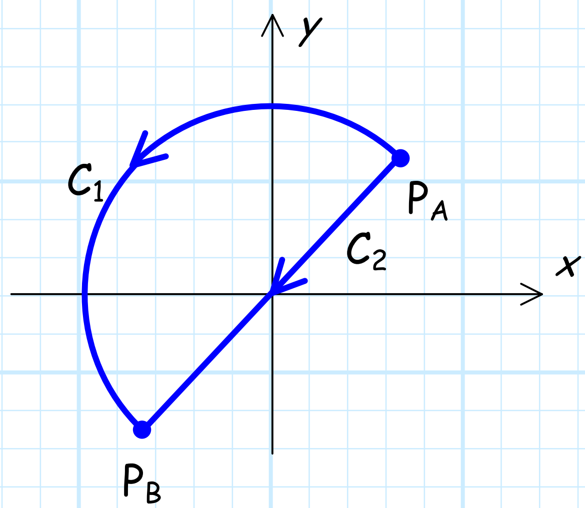
The Conservative Vector Field

Of all possible vector fields $\mathbf{A}(\bar{\mathbf{r}})$, there is a subset of vector fields called **conservative** fields. A conservative vector field is a vector field that can be expressed as the **gradient** of some scalar field $g(\bar{\mathbf{r}})$:

$$\mathbf{C}(\bar{\mathbf{r}}) = \nabla g(\bar{\mathbf{r}})$$

In other words, the gradient of **any** scalar field **always** results in a conservative field !

As we discussed earlier, a conservative field has the interesting property that its line integral is dependent on the **beginning** and **ending** points of the contour **only**! In other words, for the two contours:



we find that:

$$\int_{C_1} \mathbf{C}(\bar{r}) \cdot \overline{d\ell} = \int_{C_2} \mathbf{C}(\bar{r}) \cdot \overline{d\ell}$$

We therefore say that the line integral of a conservative field **is path independent**.

This path independence is evident when considering the **integral identity**:

$$\int_C \nabla g(\bar{r}) \cdot \overline{d\ell} = g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A)$$

where position vector $\bar{r}_B$ denotes the **ending** point (P_B) of contour C, and $\bar{r}_A$ denotes the **beginning** point (P_A). Likewise, $g(\bar{r} = \bar{r}_B)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_B$, and $g(\bar{r} = \bar{r}_A)$ denotes the value of scalar field $g(\bar{r})$ evaluated at the point denoted by $\bar{r}_A$.

Note for **one** dimension, the above identity simply reduces to the familiar expression:

$$\int_{x_a}^{x_b} \frac{\partial g(x)}{\partial x} dx = g(x = x_b) - g(x = x_a)$$

Since **every** conservative field can be written in terms of the **gradient** of a scalar field, we can use this identity to conclude:

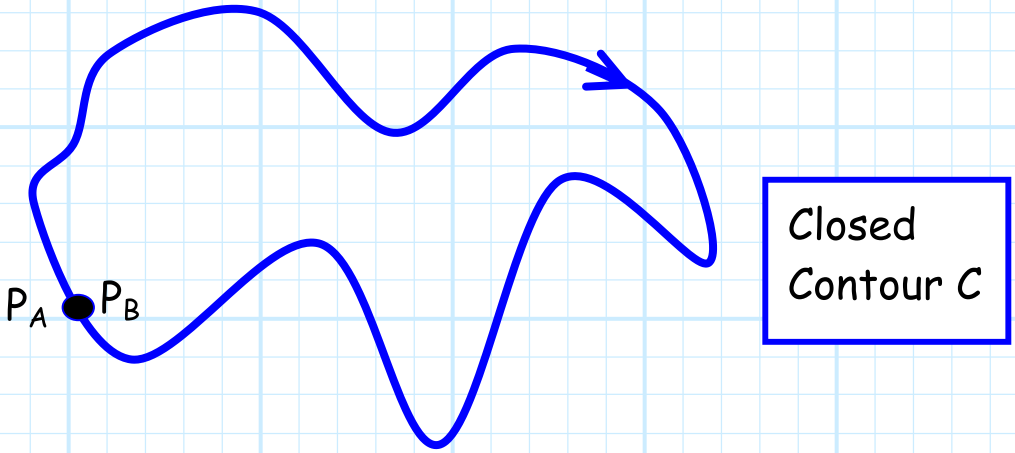
$$\begin{aligned}\int_C C(\bar{r}) \cdot \overline{d\ell} &= \int_C \nabla g(\bar{r}) \cdot \overline{d\ell} \\ &= g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A)\end{aligned}$$

Thus, the line integral **only** depends on the value $g(\bar{r})$ at the beginning and end points of a contour, the **path** taken to connect these points makes **no** difference!

Consider then what happens then if we integrate over a **closed** contour.

Q: *What the heck is a closed contour ??*

A: A closed contour is a contour whose beginning and ending is the **same** point! E.G.,



* A contour that is **not** closed is refered to as an **open** contour.

* Integration over a closed contour is **denoted** as:

$$\oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

* The integration of a **conservative** field over a **closed** contour is therefore:

$$\begin{aligned}\oint_C C(\bar{r}) \cdot \overline{d\ell} &= \oint_C \nabla g(\bar{r}) \cdot \overline{d\ell} \\ &= g(\bar{r} = \bar{r}_B) - g(\bar{r} = \bar{r}_A) \\ &= 0\end{aligned}$$

This result is due to the fact that $\bar{r}_A = \bar{r}_B$, therefore;

$$g(\bar{r} = \bar{r}_A) = g(\bar{r} = \bar{r}_B)$$

and thus the **subtraction** of these two values is **always zero**!

Let's ***summarize*** *what we know about a* ***conservative*** *vector field:*

1. A conservative vector field can always be expressed as the **gradient** of a **scalar** field.
2. The gradient of **any** scalar field is therefore a conservative vector field.
3. Integration over an **open** contour is dependent **only** on the value of scalar field $g(\bar{r})$ at the beginning and ending points of the contour (i.e., integration is **path independent**).
4. Integration of a conservative vector field over any **closed** contour is always equal to **zero**.

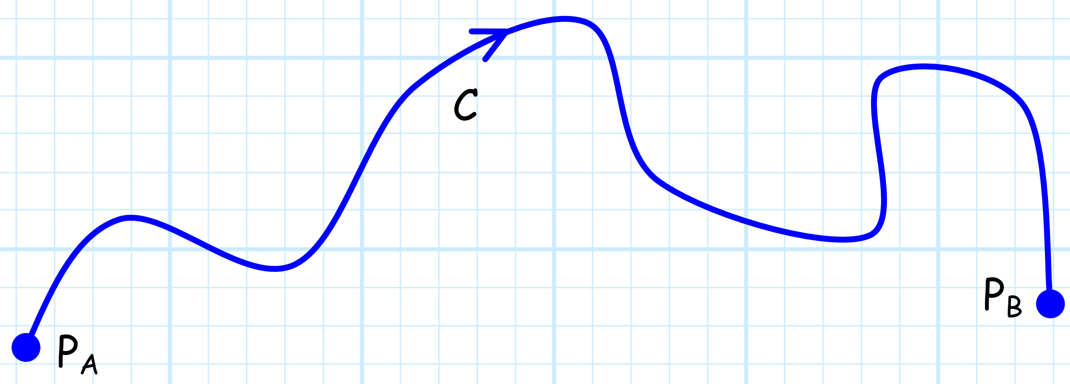
Example: Line Integrals of Conservative Fields

Consider the vector field $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$.

Evaluate the contour integral:

$$\int_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

where $\mathbf{A}(\bar{r}) = \nabla\left(x^2 + y^2\right)z$, and contour C is:



The **beginning** of contour C is the point denoted as:

$$\bar{r}_A = 3\,\hat{a}_x - \hat{a}_y + 4\,\hat{a}_z$$

while the **end** point is denote with position vector:

$$\bar{r}_B = -3\,\hat{a}_x - 2\,\hat{a}_z$$

Note that ordinarily, this would be an **impossible** problem for **us** to do!

But, we note that vector field $\mathbf{A}(\bar{r})$ is **conservative,** therefore:

$$\begin{aligned}\int_C A(\bar{r})\cdot\overline{d\ell} &= \int_C \nabla g(\bar{r})\cdot\overline{d\ell} \\ &= g(\bar{r}=\bar{r}_B)-g(\bar{r}=\bar{r}_A)\end{aligned}$$

For this problem, it is evident that:

$$g(\bar{r}) = \left(x^2+y^2\right)z$$

Therefore, $g(\bar{r}=\bar{r}_A)$ is the **scalar** field evaluated at $x = 3$, $y = -1$, $z = 4$; while $g(\bar{r}=\bar{r}_B)$ is the **scalar** field evaluated at at $x = -3$, $y = 0$, $z = -2$.

$$g(\bar{r}=\bar{r}_A) = \left((3)^2+(-1)^2\right)4 = 40$$

$$g(\bar{r}=\bar{r}_B) = \left((-3)^2+(0)^2\right)(-2) = -18$$

Therefore:

$$\begin{aligned}\int_C A(\bar{r})\cdot\overline{d\ell} &= \int_C \nabla g(\bar{r})\cdot\overline{d\ell} \\ &= g(\bar{r}=\bar{r}_B)-g(\bar{r}=\bar{r}_A) \\ &= -18-40 \\ &= -58\end{aligned}$$

The Divergence of a Vector Field

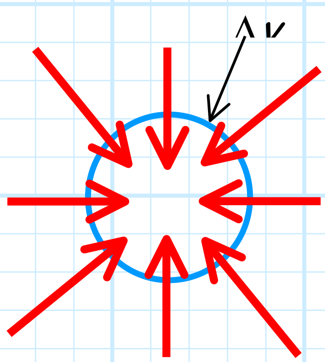
The **mathematical** definition of divergence is:

$$\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) = \lim_{\Delta v \to 0} \frac{\oiint_S \mathbf{A}(\bar{\mathbf{r}}) \cdot \overline{ds}}{\Delta v}$$

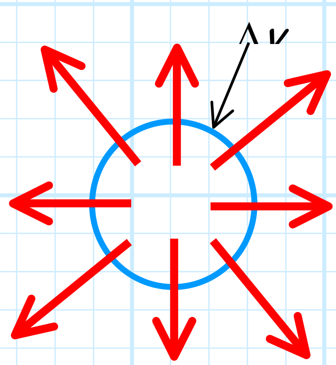
where the surface S is a **closed** surface that **completely** surrounds a **very small** volume Δv at point $\bar{\mathbf{r}}$, and where $\overline{ds}$ points **outward** from the closed surface.

From the definition of surface integral, we see that divergence basically indicates the amount of vector field $\mathbf{A}(\bar{\mathbf{r}})$ that is **converging to**, or **diverging from**, a given point.

For example, consider these vector fields in the region of a **specific point**:



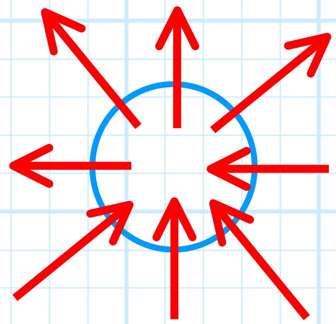
$\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) < 0$



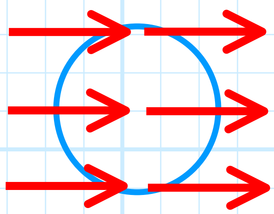
$\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) > 0$

The field on the left is **converging** to a point, and therefore the divergence of the vector field at that point is **negative**. Conversely, the vector field on the right is **diverging** from a point. As a result, the divergence of the vector field at that point is **greater than zero**.

Consider some **other** vector fields in the region of a specific point:



$\nabla \cdot \mathbf{A}(\bar{r}) = 0$



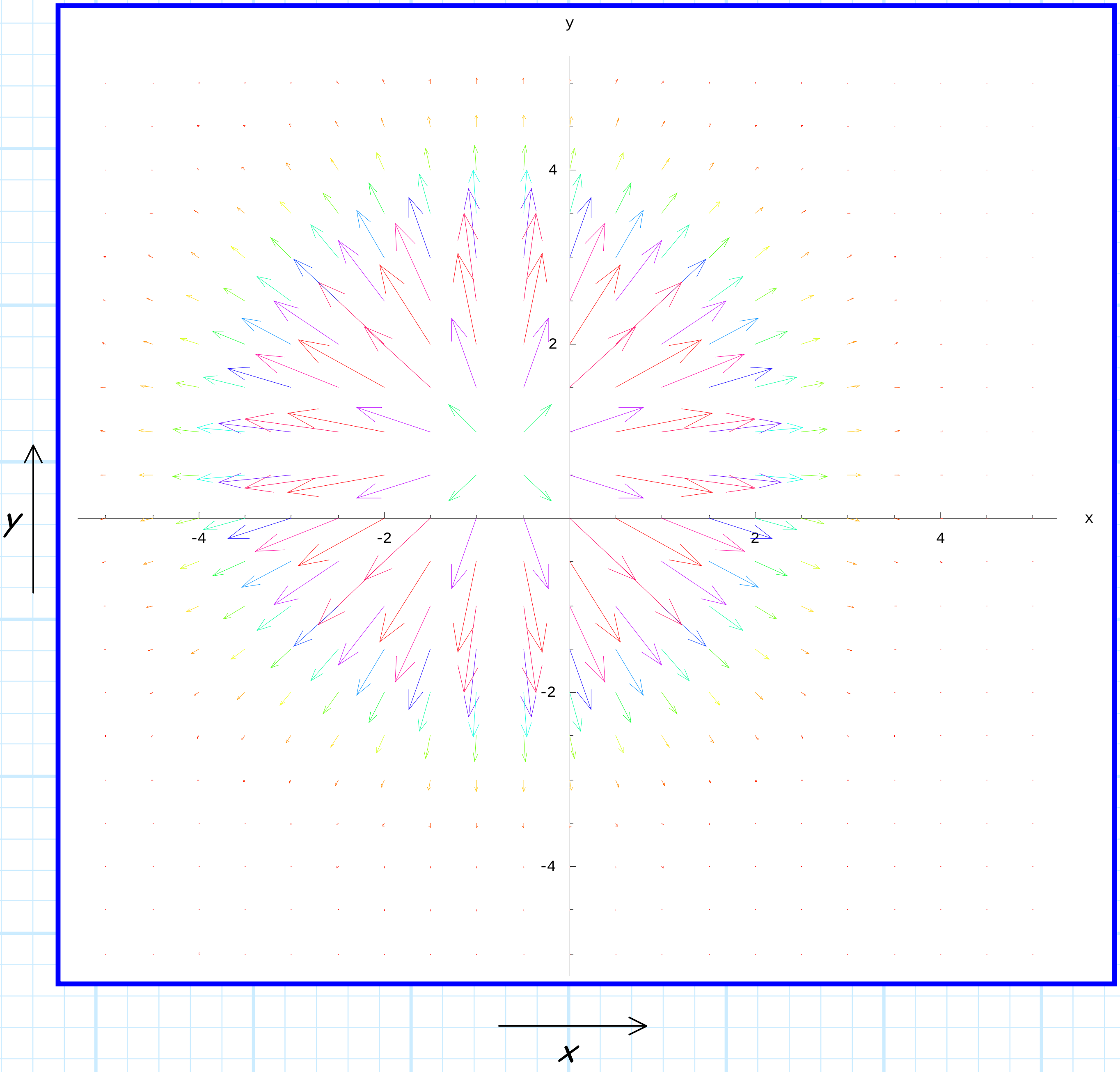
$\nabla \cdot \mathbf{A}(\bar{r}) = 0$

For each of these vector fields, the surface integral is **zero**. Over some portions of the surface, the normal component is positive, whereas on other portions, the normal component is negative. However, **integration** over the entire surface is equal to zero—the divergence of the vector field at this point is zero.

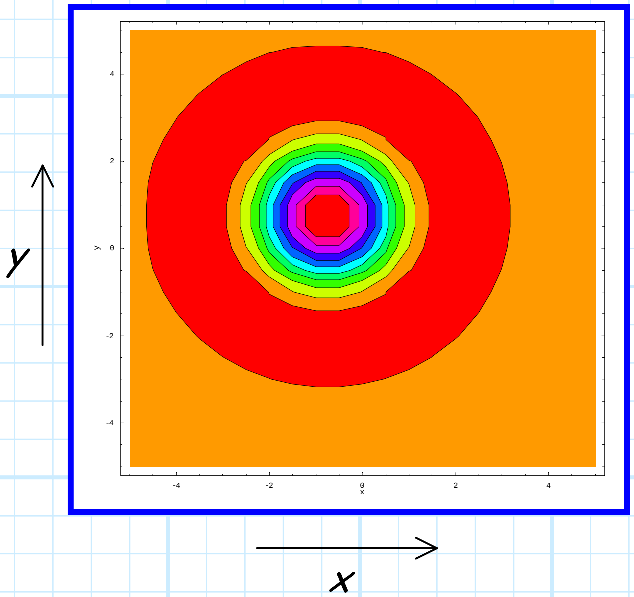
* **Generally**, the divergence of a vector field results in a scalar field (divergence) that is positive in some regions in space, negative other regions, and zero elsewhere.

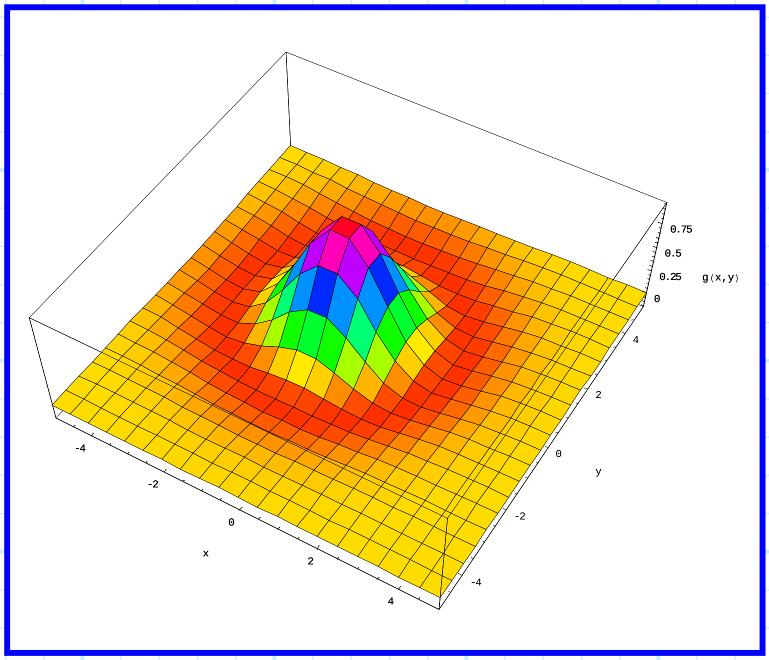
* For most **physical** problems, the divergence of a vector field provides a scalar field that represents the **sources** of the vector field.

For example, consider this two-dimensional vector field $\mathbf{A}(x,y)$, plotted on the x,y plane:



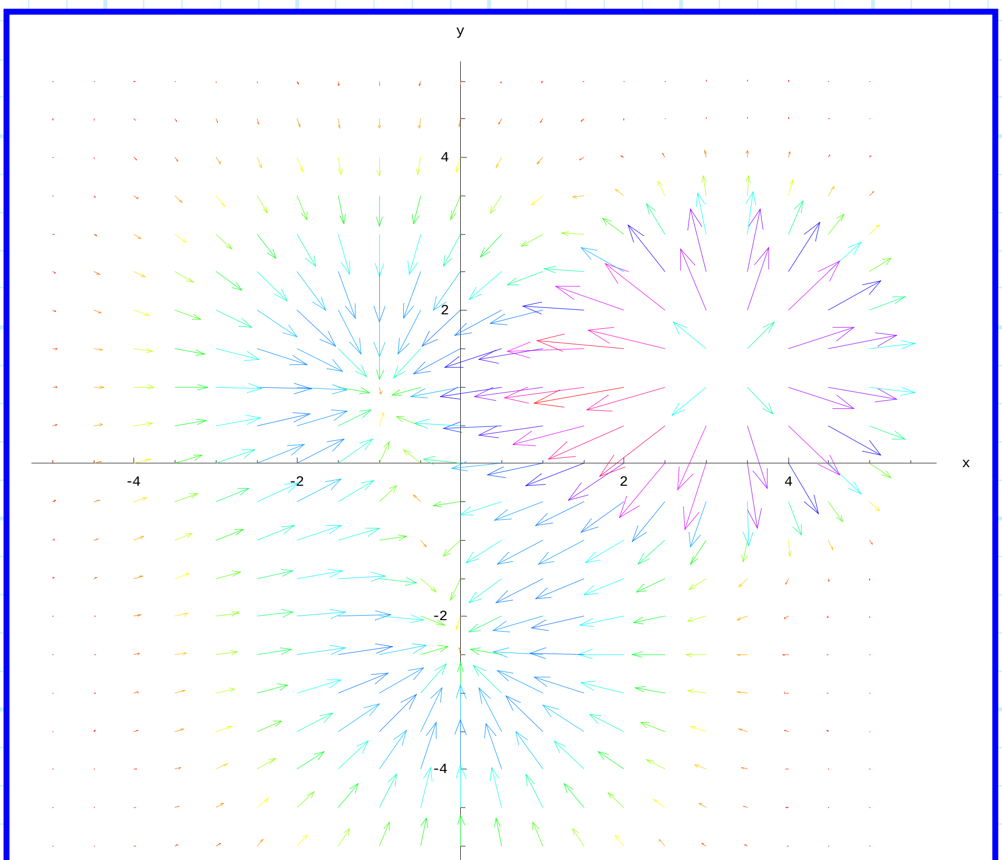
We can take the divergence of this vector field, resulting in the scalar field $g(x,y) = \nabla \cdot \mathbf{A}(x,y)$. Plotting this scalar function on the x,y plane:



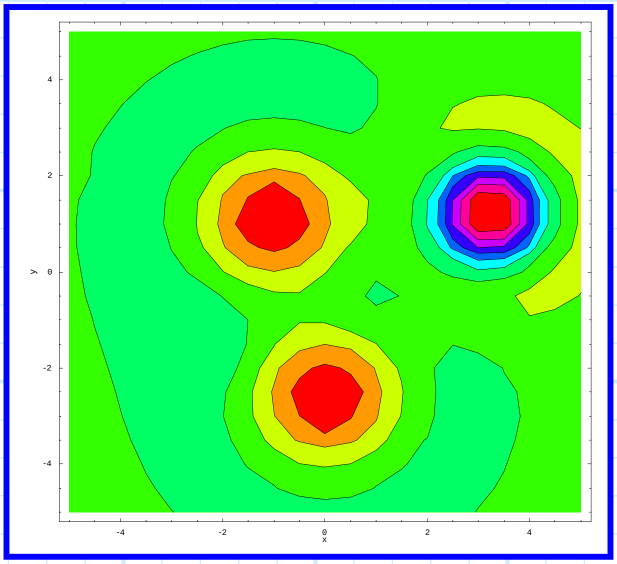


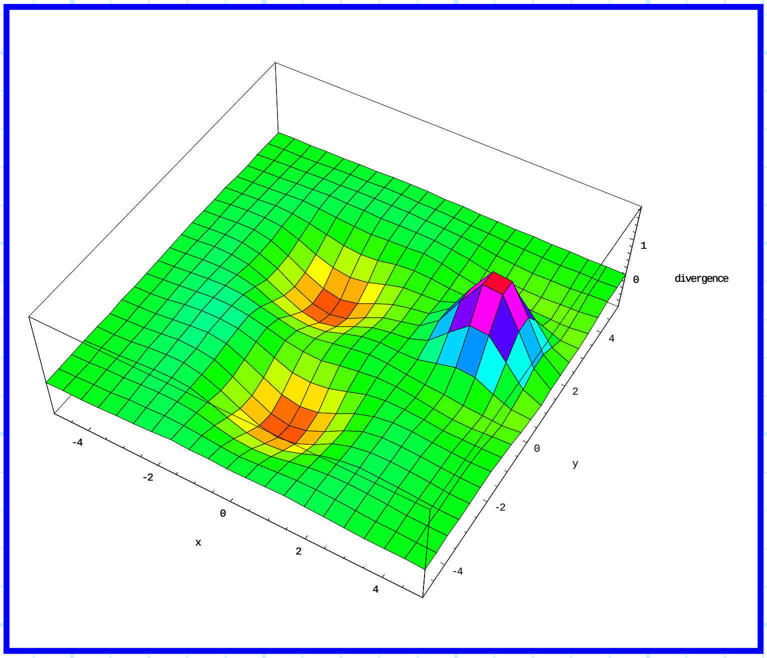
Both plots indicate that the divergence is largest in the vicinity of point x=-1, y=1. However, notice that the value of g(x,y) is non-zero (both positive and negative) for most points (x,y).

Consider now this vector field:

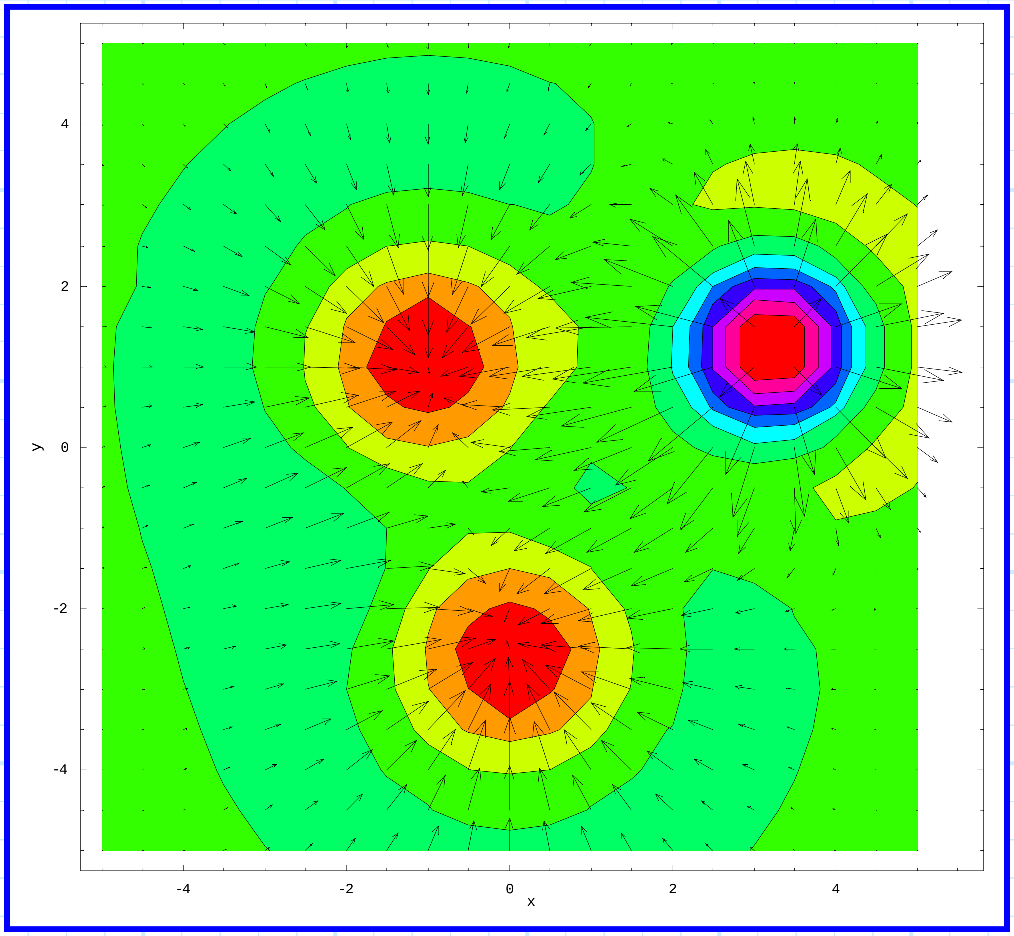


The **divergence** of this vector field is the **scalar** field:





Combining the vector field and scalar field plots, we can examine the **relationship** between each:



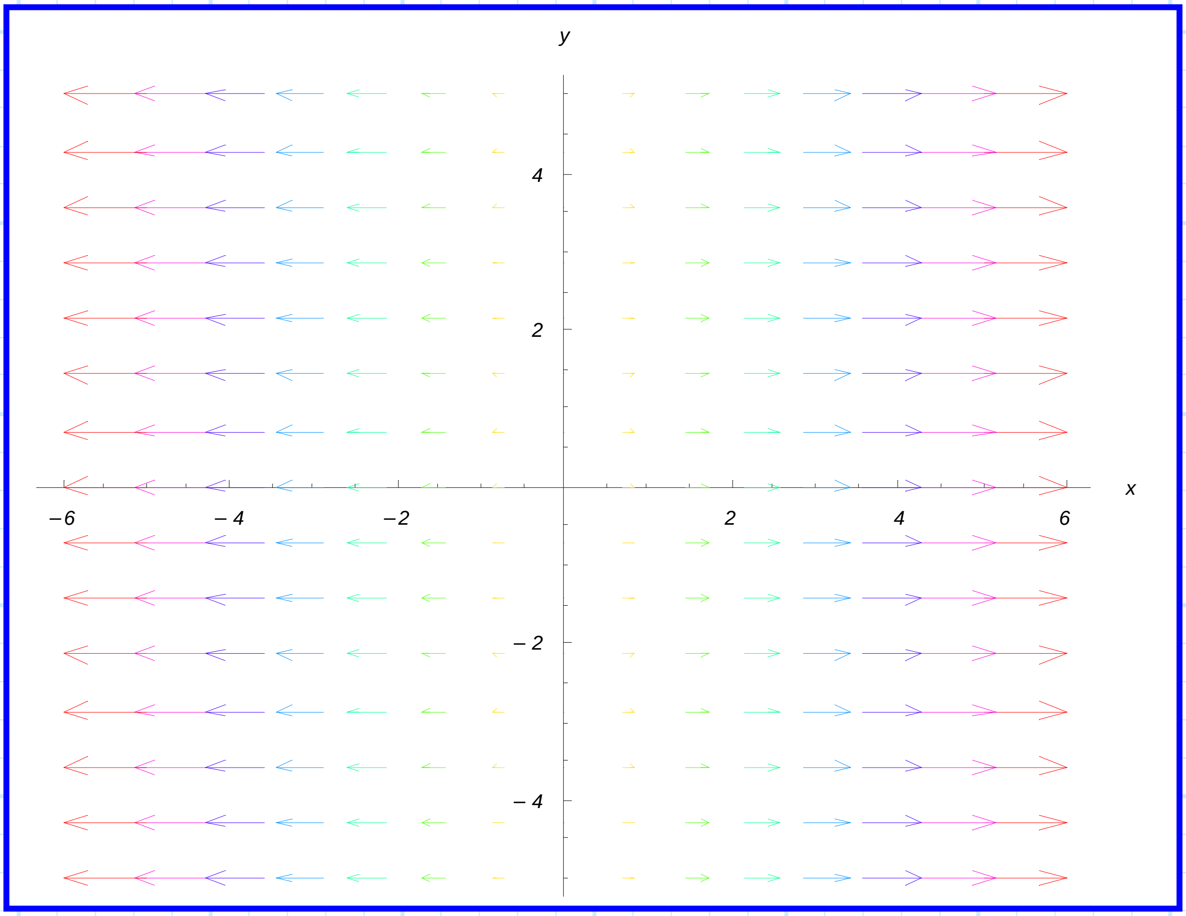
Look closely! Although the relationship between the scalar field and the vector field may appear at first to be the **same** as with the **gradient** operator, the two relationships are **very** different.

Remember:

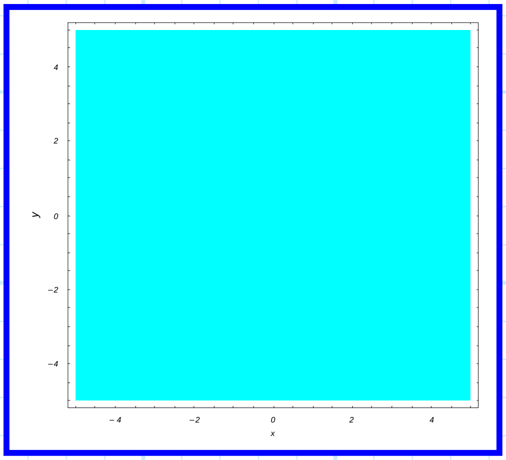
a) **gradient** produces a **vector** field that indicates the change in the original **scalar** field, whereas:

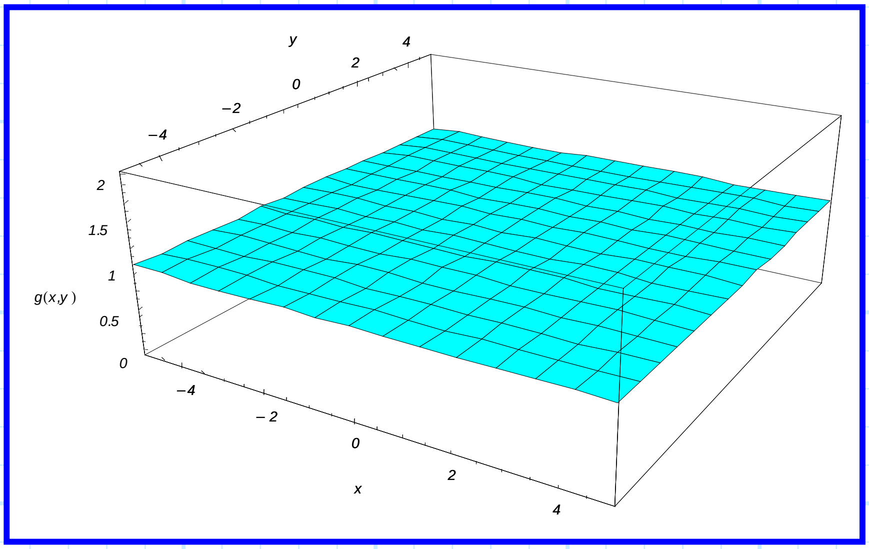
b) **divergence** produces a **scalar** field that indicates some change (i.e., divergence or convergence) of the original **vector** field.

The divergence of **this** vector field is interesting—it steadily increases as we move away from the *y*-axis.

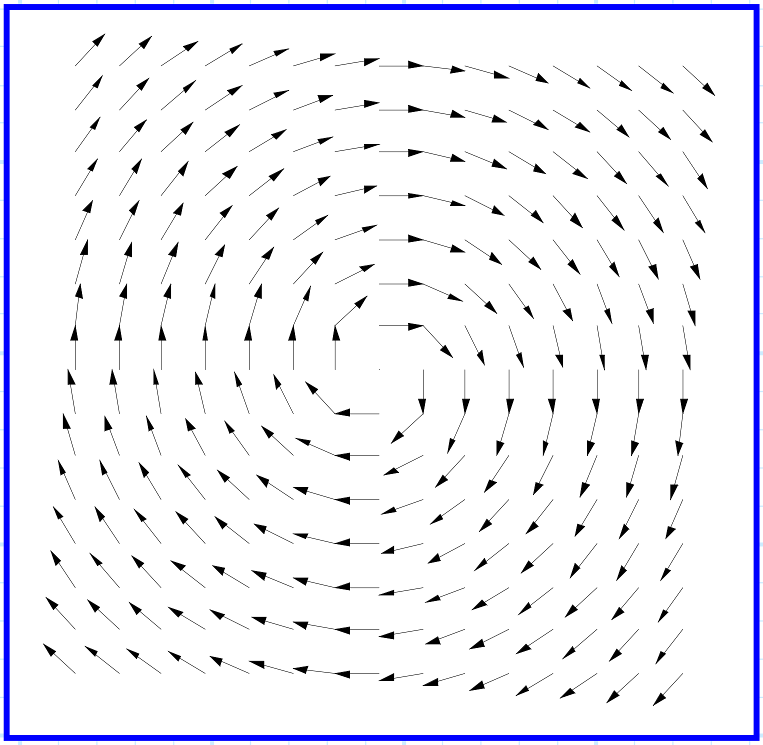


Yet, the divergence of this vector field produces a scalar field equal to one—**everywhere** (i.e., a **constant** scalar field)!

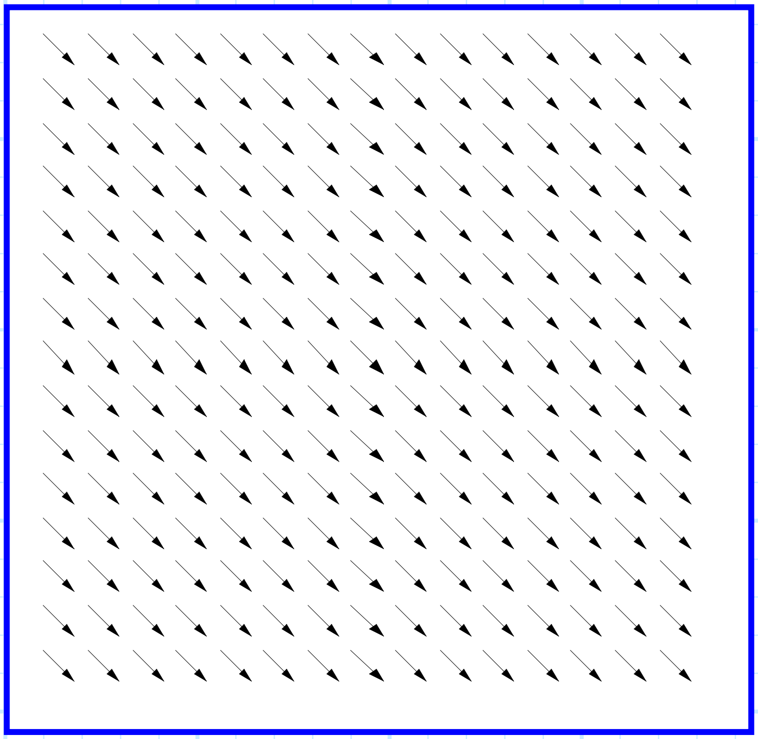




Likewise, note the divergence of these vector fields—it is **zero** at all points (x,y):



$$\nabla \cdot \mathbf{A}(x,y) = 0$$



$$\nabla \cdot \mathbf{A}(x,y) = 0$$

Although the examples we have examined here were all **two**-dimensional, keep in mind that both the original vector field, as well as the scalar field produced by divergence, will typically be **three-dimensional**!

The Divergence in Coordinate Systems

Consider now the **divergence** of vector fields expressed with our **coordinate systems**:

Cartesian

$$\nabla \cdot \mathbf{A}(\bar{\mathrm{r}}) = \frac{\partial A_x(\bar{r})}{\partial x} + \frac{\partial A_y(\bar{r})}{\partial y} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Cylindrical

$$\nabla \cdot \mathbf{A}(\bar{r}) = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho(\bar{r})\right)}{\partial \rho}\right] + \frac{1}{\rho}\frac{\partial A_\phi(\bar{r})}{\partial \phi} + \frac{\partial A_z(\bar{r})}{\partial z}$$

Spherical

$$\nabla \cdot \mathbf{A}(\bar{r}) = \frac{1}{r^2}\left[\frac{\partial\left(r^2 A_r(\bar{r})\right)}{\partial r}\right] + \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta\, A_\theta(\bar{r})\right)}{\partial \theta}\right] + \frac{1}{r\sin\theta}\frac{\partial A_\phi(\bar{r})}{\partial \phi}$$

Note that, as with the gradient expression, the divergence expressions for cylindrical and spherical coordinate systems are more **complex** than those of Cartesian. Be **careful** when you use these expressions!

For **example**, consider the vector field:

$$\mathbf{A}(\bar{\mathbf{r}}) = \frac{\sin\theta}{r}\hat{a}_r$$

Therefore, $A_\theta = 0$ and $A_\phi = 0$, leaving:

$$\begin{aligned}
\nabla \cdot \mathbf{A}(\bar{\mathbf{r}}) &= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 A_r(\bar{r})\right)\right] \\
&= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\sin\theta}{r}\right)\right] \\
&= \frac{1}{r^2}\left[\frac{\partial\left(r\sin\theta\right)}{\partial r}\right] \\
&= \frac{1}{r^2}[\sin\theta] = \frac{\sin\theta}{r^2}
\end{aligned}$$

The Divergence Theorem

Recall we studied volume integrals of the form:

$$\iiint_V g(\bar{r})\,dv$$

It turns out that **any** and **every** scalar field can be written as the divergence of some **vector** field, i.e.:

$$g(\bar{r}) = \nabla \cdot \mathbf{A}(\bar{r})$$

Therefore we can equivalently write any volume integral as:

$$\iiint_V \nabla \cdot \mathbf{A}(\bar{r})\,dv$$

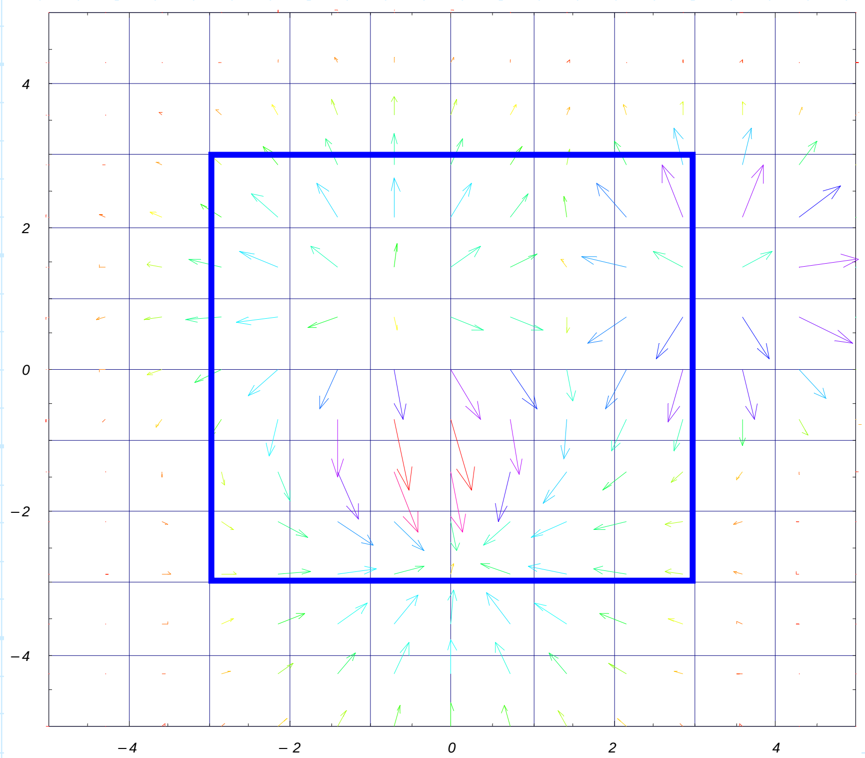
The **divergence theorem** states that this integral is equal to:

$$\iiint_V \nabla \cdot \mathbf{A}(\bar{r})\,dv = \oiint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

where S is the **closed** surface that completely surrounds volume V, and vector $\overline{ds}$ points **outward** from the closed surface. For example, if volume V is a **sphere**, then S is the **surface** of that sphere.

The divergence theorem states that the **volume** integral of a scalar field can be likewise evaluated as a **surface** integral of a vector field!

What the divergence theorem indicates is that the **total** "divergence" of a vector field through the **surface** of any volume is equal to the sum (i.e., integration) of the divergence at **all points** within the **volume**.



In other words, if the vector field is **diverging** from some point in the volume, it must simultaneously be **converging** to another adjacent point within the volume—the net effect is therefore **zero**!

Thus, the only values that make **any** difference in the **volume integral** are the divergence or convergence of the vector field across the surface surrounding the volume—vectors that will be converging or diverging to adjacent points **outside** the volume (across the surface) from points **inside** the volume. Since these points just outside the volume are not included in the integration, their net effect is **non-zero**!

The Curl of a Vector Field

Say $\nabla \times \mathbf{A}(\bar{\mathrm{r}}) = \mathbf{B}(\bar{\mathrm{r}})$. The **mathematical** definition of Curl is given as:

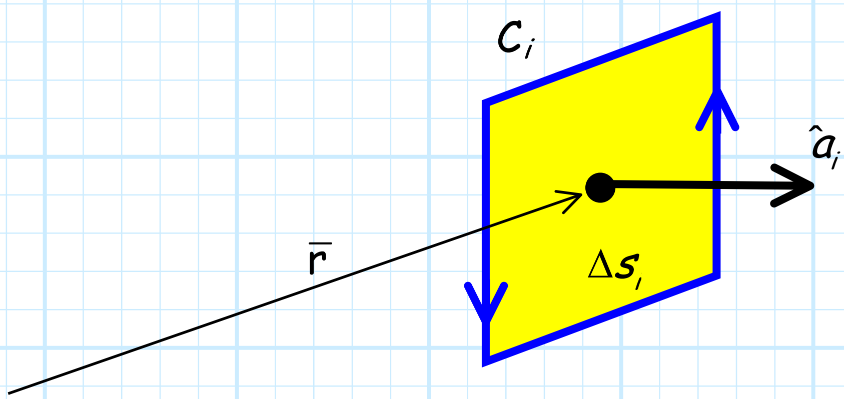
$$B_i(\bar{\mathrm{r}}) = \lim_{\Delta s \to 0} \frac{\oint_{C_i} \mathbf{A}(\bar{\mathrm{r}}) \cdot \overline{d\ell}}{\Delta s_i}$$

This rather complex equation requires some **explanation** !

* $B_i(\bar{\mathrm{r}})$ is the scalar component of vector $\mathbf{B}(\bar{\mathrm{r}})$ in the direction defined by unit vector $\hat{a}_i$ (e.g., $\hat{a}_x, \hat{a}_\rho, \hat{a}_\theta$).

* The small surface Δs_i is centered at point $\bar{\mathrm{r}}$, and oriented such that it is normal to unit vector $\hat{a}_i$.

* The contour C_i is the closed contour that surrounds surface Δs_i.



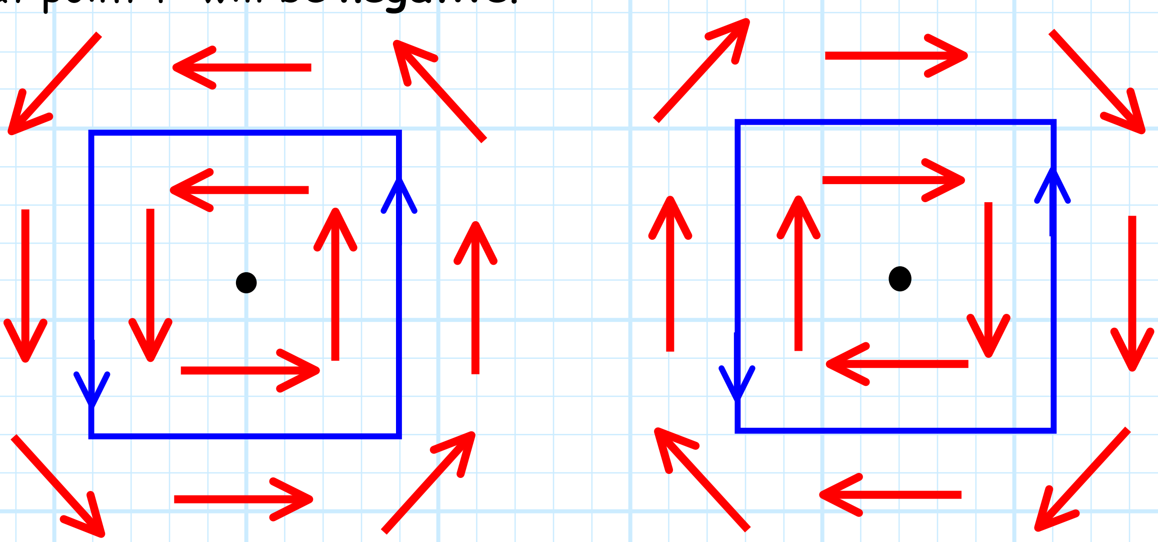
Note that this derivation must be completed for **each** of the **three** orthonormal base vectors in order to completely define $\mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$.

Q: *What does curl tell us ?*

A: Curl is a measurement of the **circulation** of vector field $\mathbf{A}(\bar{r})$ around point $\bar{r}$.

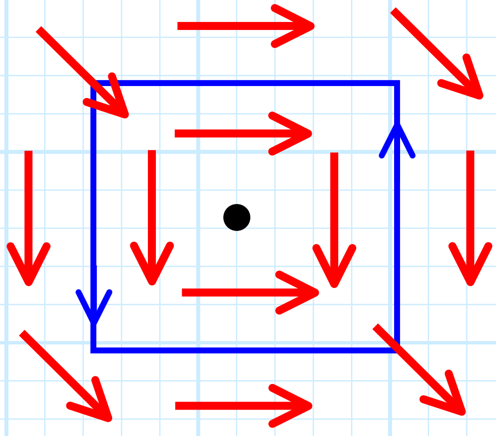
If a component of vector field $\mathbf{A}(\bar{r})$ is pointing in the direction $\overline{d\ell}$ at every point on contour C_i (i.e., **tangential** to the contour). Then the line integral, and thus the curl, will be **positive**.

If, however, a component of vector field $\mathbf{A}(\bar{r})$ points in the opposite direction (-$\overline{d\ell}$) at every point on the contour, the curl at point $\bar{r}$ will be **negative**.

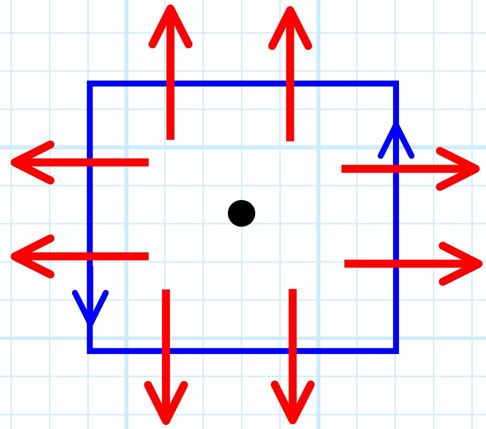


$B_i > 0$ $B_i < 0$

Likewise, **these** vector fields will result in a curl with **zero** value at point $\bar{r}$:



$B_i = 0$

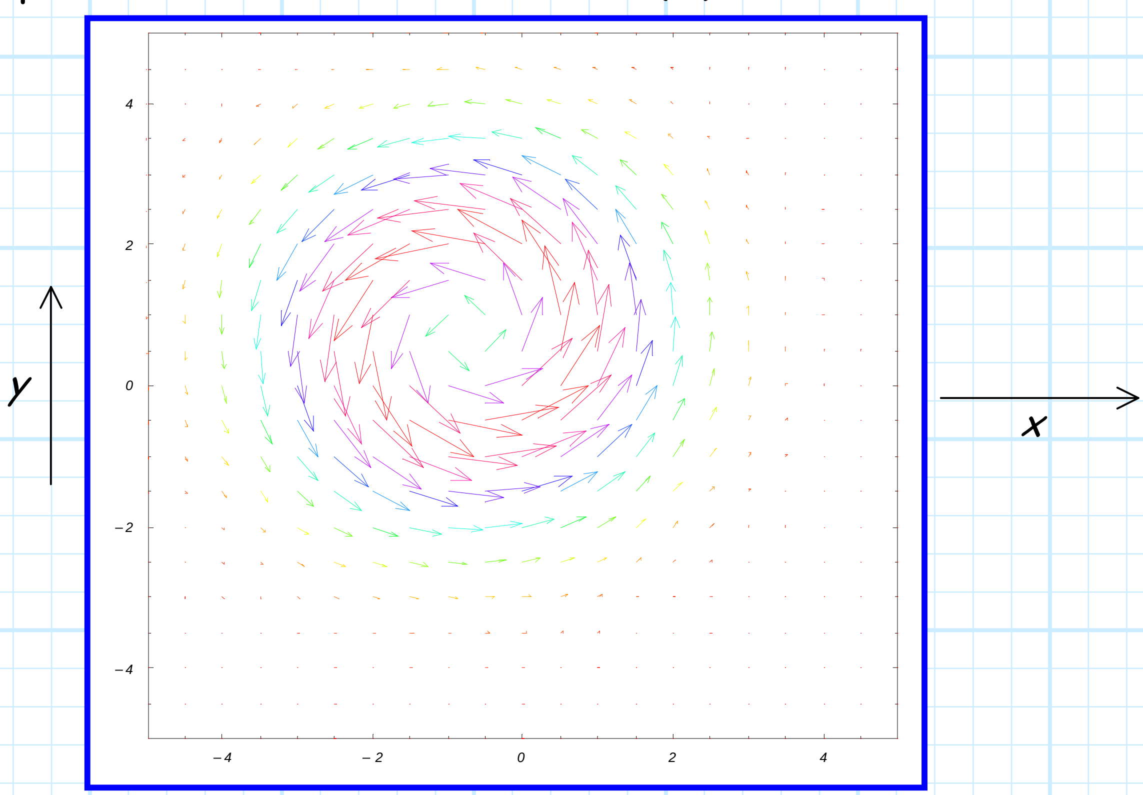


$B_i = 0$

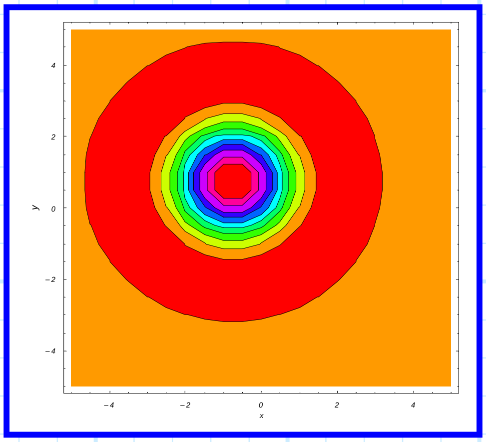
* **Generally**, the curl of a vector field result is in another vector field whose magnitude is positive in some regions of space, negative in other regions, and zero elsewhere.

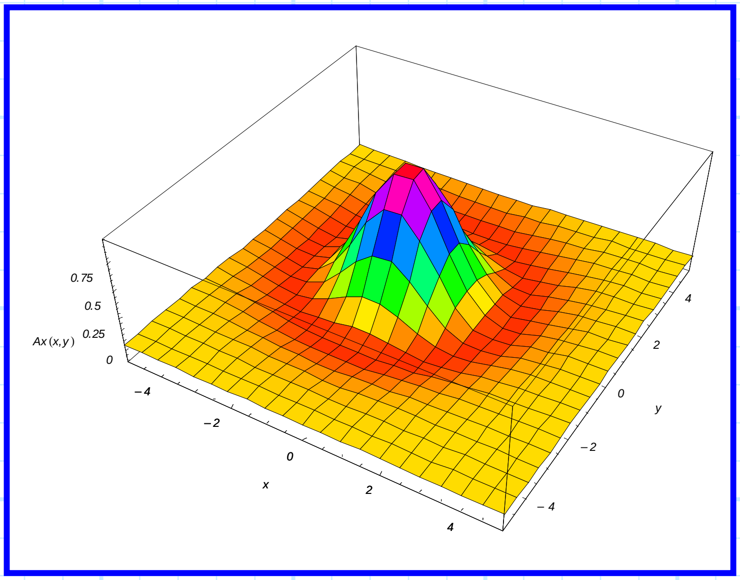
* For most **physical** problems, the curl of a vector field provides another vector field that indicates **rotational sources** (i.e., "paddle wheels") of the original vector field.

For example, consider this vector field $\mathbf{A}(\bar{r})$:

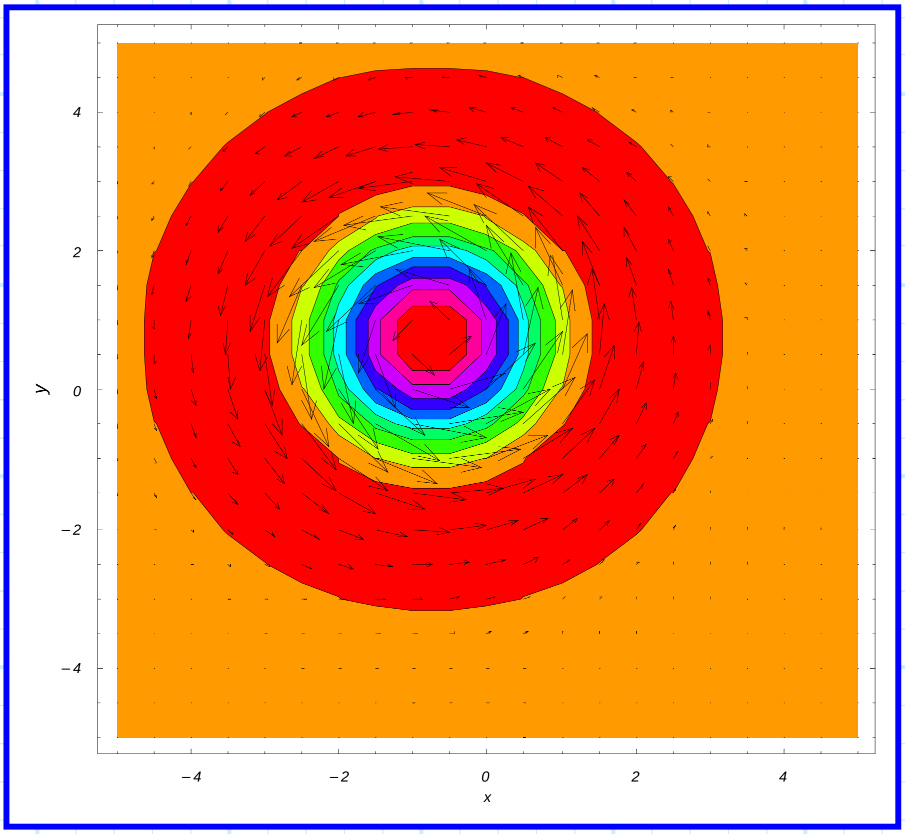


If we take the curl of $\mathbf{A}(\bar{r})$, we get a **vector field** which points in the direction $\hat{a}_z$ at **all** points (x,y). The **scalar component** of this resulting vector field (i.e., $B_z(\bar{r})$) is:



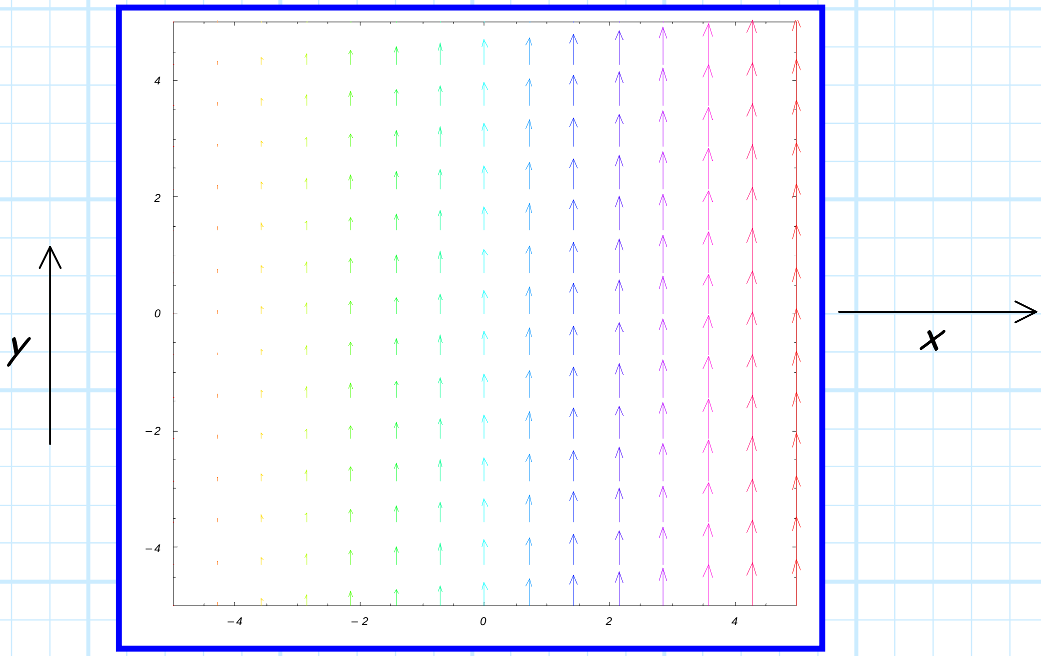


The relationship between the original vector field $\mathbf{A}(\bar{r})$ and its resulting curl perhaps is best shown when plotting both together:

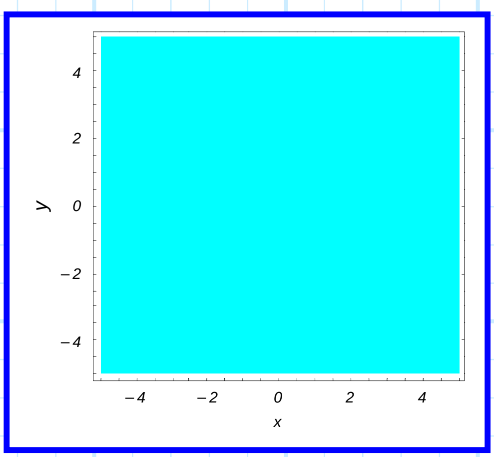


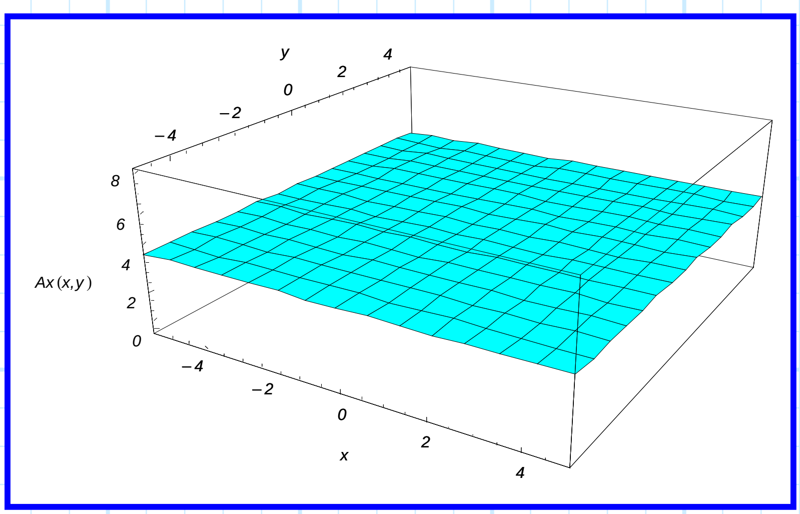
Note this **scalar** component is **largest** in the region near point x=-1, y=1, indicating a "**rotational source**" in this region. This is likewise apparent from the original plot of vector field $\mathbf{A}(\bar{r})$.

Consider now another **vector field**:

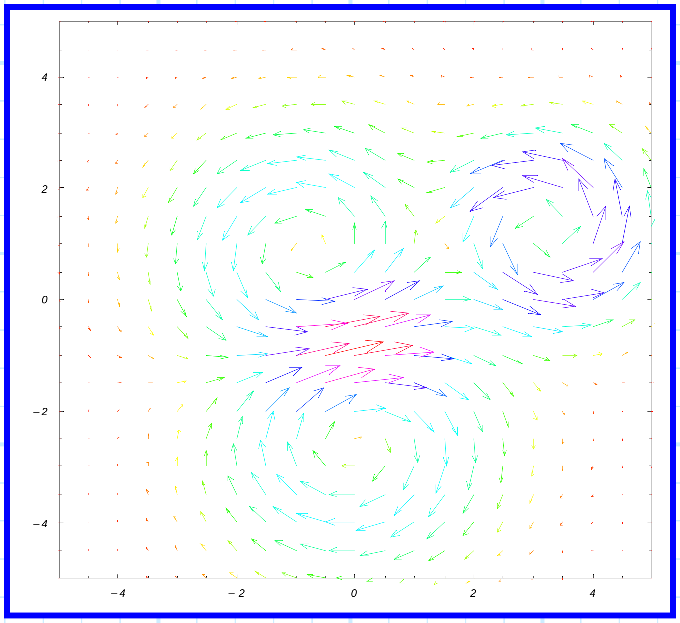


Although at first this vector field **appears** to exhibit no rotation, it in fact has a **non-zero** curl at **every** point ($\mathbf{B}(\bar{r}) = 4.0\ \hat{\mathbf{a}}_z$) ! Again, the direction of the resulting field is in the direction $\hat{a}_z$. We plot therefore the **scalar** component in this direction (i.e., $B_z(\bar{r})$):

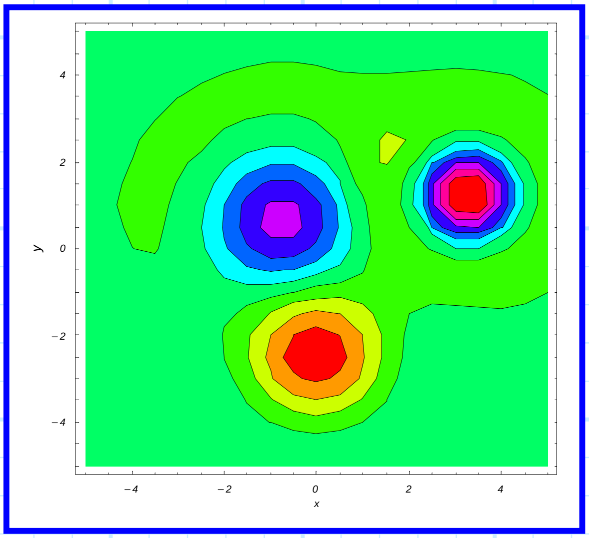


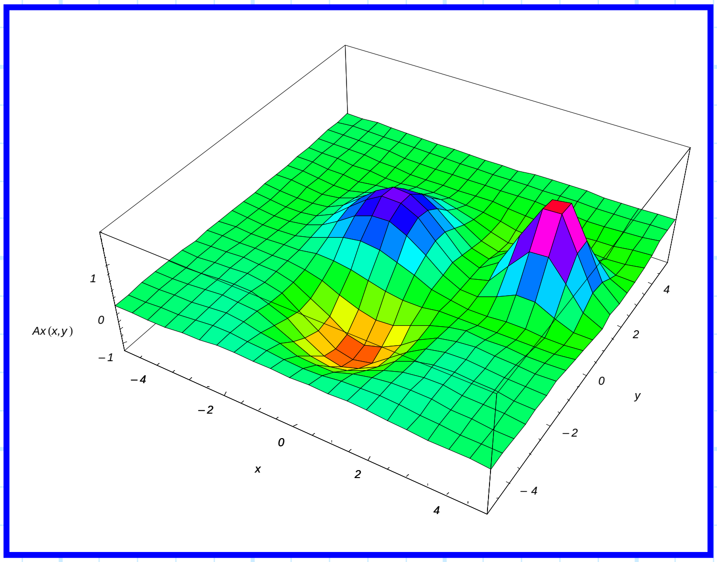


We might encounter a more **complex** vector field, such as:

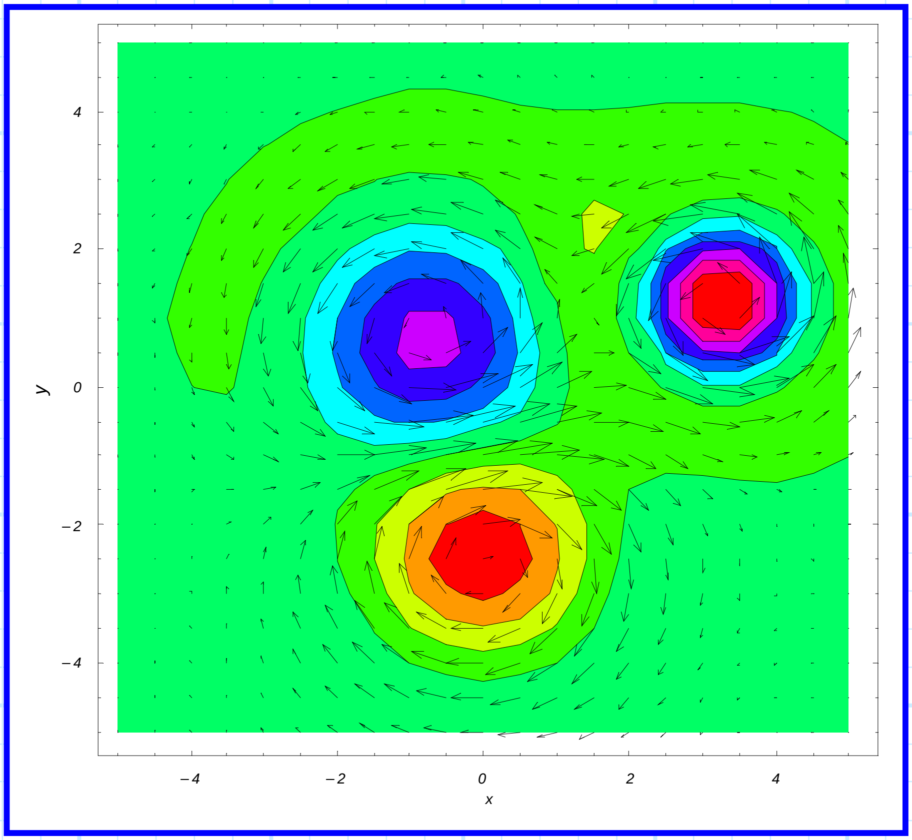


If we take the **curl** of this vector field, the resulting vector field will **again** point in the direction $\hat{a}_z$ at every point (i.e., $B_x(\bar{r}) = B_y(\bar{r}) = 0$). Plotting therefore the scalar component of the resulting vector field (i.e., $B_z(\bar{r})$), we get:

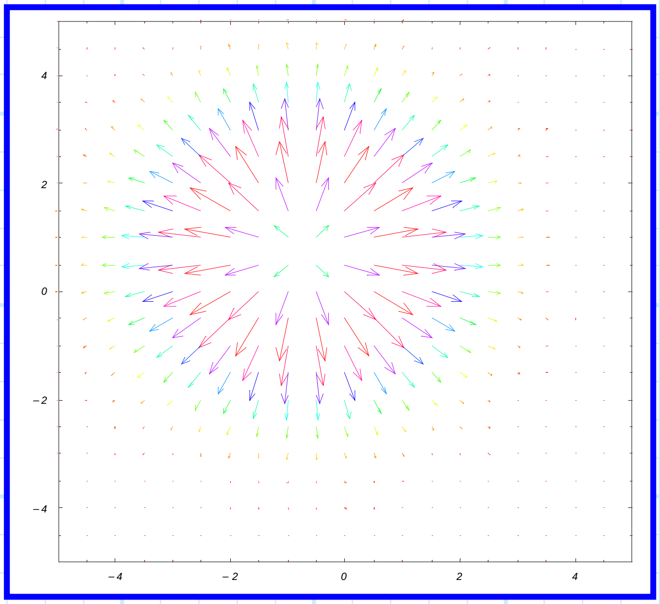


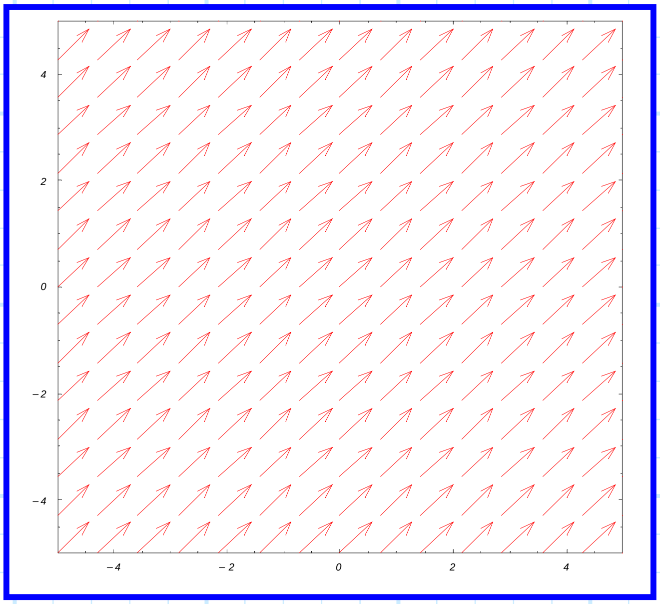


Note these plots indicate that there are **two** regions of large **counter** clockwise rotation in the original vector field, and **one** region of large **clockwise** rotation.



Finally, consider **these** vector fields:





The curl of these vector fields is **zero** at all points. It is apparent that there is no **rotation** in either of these vector fields!

Curl in Coordinate Systems

Consider now the curl of vector fields expressed using our coordinate systems.

Cartesian

$$\nabla \times \mathbf{A}(\bar{r}) = \left[\frac{\partial A_y(\bar{r})}{\partial z} - \frac{\partial A_z(\bar{r})}{\partial y}\right]\hat{a}_x + \left[\frac{\partial A_z(\bar{r})}{\partial x} - \frac{\partial A_x(\bar{r})}{\partial z}\right]\hat{a}_y + \left[\frac{\partial A_x(\bar{r})}{\partial y} - \frac{\partial A_y(\bar{r})}{\partial x}\right]\hat{a}_z$$

Cylindrical

$$\nabla \times \mathbf{A}(\bar{r}) = \left[\frac{1}{\rho}\frac{\partial A_z(\bar{r})}{\partial \phi} - \frac{\partial A_\phi(\bar{r})}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial A_\rho(\bar{r})}{\partial z} - \frac{\partial A_z(\bar{r})}{\partial \rho}\right]\hat{a}_\phi + \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\phi(\bar{r})\right) - \frac{1}{\rho}\frac{\partial A_\rho(\bar{r})}{\partial \phi}\right]\hat{a}_z$$

Spherical

$$\nabla \times \mathbf{A}(\bar{r}) = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi(\bar{r})\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta(\bar{r})}{\partial\phi}\right]\hat{a}_r$$
$$+\left[\frac{1}{r\sin\theta}\frac{\partial A_r(\bar{r})}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi(\bar{r})\right)\right]\hat{a}_\theta$$
$$+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta(\bar{r})\right) - \frac{1}{r}\frac{\partial A_r(\bar{r})}{\partial\theta}\right]\hat{a}_\phi$$

Yikes! These expressions are **very** complex. Precision, organization, and patience are required to **correctly** evaluate the **curl** of a vector field !

Stokes' Theorem

Consider a vector field $\mathbf{B}(\bar{r})$ where:

$$\mathbf{B}(\bar{r}) = \nabla x \mathbf{A}(\bar{r})$$

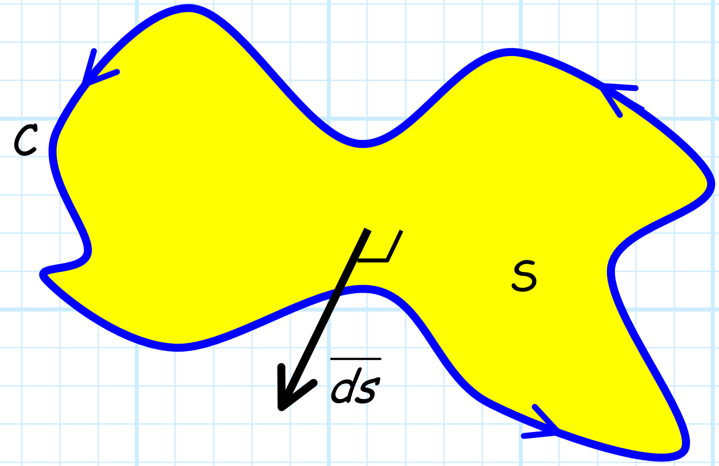
Say we wish to integrate this vector field over an **open** surface S:

$$\iint_S \mathbf{B}(\bar{r}) \cdot \overline{ds} = \iint_S \nabla x \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

We can likewise evaluate this integral using **Stokes' Theorem**:

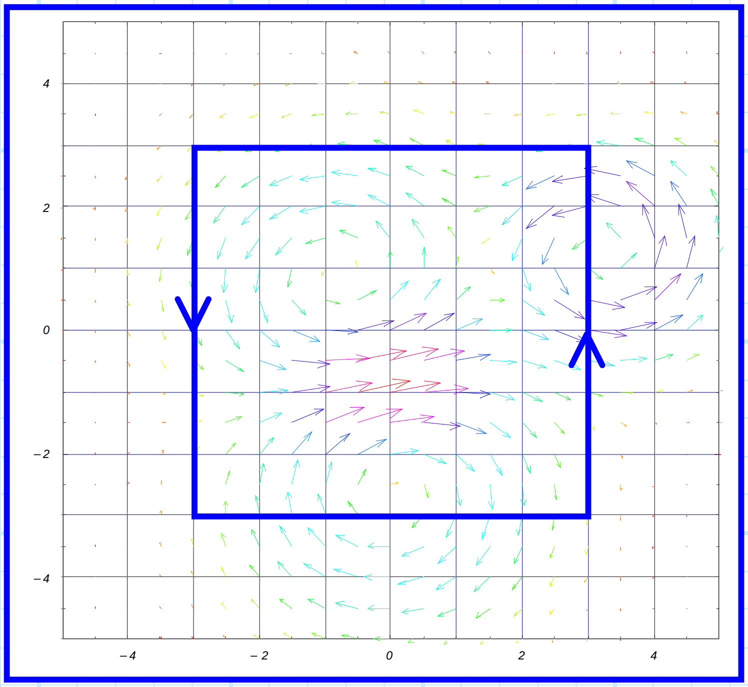
$$\iint_S \nabla x \mathbf{A}(\bar{r}) \cdot \overline{ds} = \oint_C \mathbf{A}(\bar{r}) \cdot \overline{d\ell}$$

In this case, the contour *C* is a **closed** contour that **surrounds** surface S. The direction of C is defined by $\overline{ds}$ and the **right - hand rule**. In other words *C* rotates **counter clockwise** around $\overline{ds}$. E.G.,



* Stokes' Theorem allows us to evaluate the **surface** integral of a curl as simply a **contour** integral !

* Stokes' Theorem states that the summation (i.e., integration) of the circulation at **every** point on a surface is simply the **total** "circulation" around the closed **contour** surrounding the surface.



In other words, if the vector field is **rotating counter-clockwise** around some point in the volume, it must simultaneously be **rotating clockwise** around adjacent points within the volume—the net effect is therefore **zero**!

Thus, the only values that make **any** difference in the **surface integral** is the rotation of the vector field around points that lie on the surrounding contour (i.e., the very edge of the surface S). These vectors are likewise rotating in the opposite direction around adjacent points—but these points do **not** lie on the surface (thus, they are **not** included in the integration). The net effect is therefore **non-zero**!

Note that if S is a **closed surface**, then there is **no** contour *C* that exists! In other words:

$$\oiint_S \nabla \times \mathbf{A}(\bar{r}) \cdot \overline{ds} = \oint_0 \mathbf{A}(\bar{r}) \cdot \overline{d\ell} = 0$$

Therefore, integrating the **curl of any vector field** over a **closed** surface **always** equals zero.

The Curl of Conservative Fields

Recall that every **conservative** field can be written as the gradient of some scalar field:

$$\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$$

Consider now the **curl of a conservative field**:

$$\nabla x \mathbf{C}(\bar{r}) = \nabla x \nabla g(\bar{r})$$

Recall that if $\mathbf{C}(\bar{r})$ is expressed using the **Cartesian** coordinate system, the curl of $\mathbf{C}(\bar{r})$ is:

$$\nabla x \mathbf{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

Likewise, the **gradient** of $g(\bar{r})$ is:

$$\nabla g(\bar{r}) = \mathbf{C}(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x}\hat{a}_x + \frac{\partial g(\bar{r})}{\partial y}\hat{a}_y + \frac{\partial g(\bar{r})}{\partial z}\hat{a}_z$$

Therefore:

$$C_x(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x}$$
$$C_y(\bar{r}) = \frac{\partial g(\bar{r})}{\partial y}$$
$$C_z(\bar{r}) = \frac{\partial g(\bar{r})}{\partial z}$$

Combining these two results:

$$\begin{aligned}\nabla \times \nabla g(\bar{r}) = &\left[\frac{\partial^2 g(\bar{r})}{\partial y \partial z} - \frac{\partial^2 g(\bar{r})}{\partial z \partial y}\right]\hat{a}_x \\ &+ \left[\frac{\partial^2 g(\bar{r})}{\partial z \partial x} - \frac{\partial^2 g(\bar{r})}{\partial x \partial z}\right]\hat{a}_y \\ &+ \left[\frac{\partial^2 g(\bar{r})}{\partial x \partial y} - \frac{\partial^2 g(\bar{r})}{\partial y \partial x}\right]\hat{a}_z\end{aligned}$$

Since, for example:

$$\frac{\partial^2 g(\bar{r})}{\partial y \partial z} = \frac{\partial^2 g(\bar{r})}{\partial z \partial y},$$

each component of $\nabla \times \nabla g(\bar{r})$ is then equal to **zero**, and we can say:

$$\nabla \times \nabla g(\bar{r}) = \nabla \times \mathbf{C}(\bar{r}) = 0$$

The **curl** of every **conservative** field is **equal to zero** !

Likewise, we have determined that:

$$\nabla \times \nabla g(\bar{r}) = 0$$

for **all** scalar functions $g(\bar{r})$.

Q: *Are there some **non-**conservative fields whose curl is also equal to zero?*

A: **NO!** The curl of a conservative field, and **only** a conservative field, is equal to **zero**.

Thus, we have way to **test** whether some vector field $\mathbf{A}(\bar{r})$ is conservative: **evaluate its curl**!

1. If the result **equals zero**—the vector field **is** conservative.

2. If the result is **non-zero**—the vector field **is not** conservative.

Let's again **recap** what we've learned about **conservative** fields:

1. The line integral of a conservative field is **path independent**.

2. Every conservative field can be expressed as the **gradient** of some scalar field.

3. The gradient of **any** and **all** scalar fields is a conservative field.

4. The line integral of a conservative field around any **closed** contour is equal to zero.

5. The **curl** of every conservative field is equal to **zero**.

6. The **curl** of a vector field is zero **only** if it is conservative.

The Solenoidal Vector Field

1. We of course recall that a **conservative** vector field $\mathbf{C}(\bar{r})$ can be identified from its curl, which is always equal to zero:

$$\nabla x \mathbf{C}(\bar{r}) = 0$$

Similarly, there is **another** type of vector field $\mathbf{S}(\bar{r})$, called a **solenoidal** field, whose **divergence** is always equal to zero:

$$\nabla \cdot \mathbf{S}(\bar{r}) = 0$$

Moreover, we find that **only** solenoidal vector have zero divergence! Thus, zero divergence is a **test** for determining if a given vector field is solenoidal.

We sometimes refer to a solenoidal field as a **divergenceless** field.

2. Recall that **another** characteristic of a **conservative** vector field is that it can be expressed as the **gradient** of some **scalar** field (i.e., $\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$).

Solenoidal vector fields have a **similar** characteristic! Every solenoidal vector field can be expressed as the **curl** of some other vector field (say $\mathbf{A}(\bar{r})$).

$$\mathbf{S}(\bar{r}) = \nabla x \mathbf{A}(\bar{r})$$

Additionally, we find that **only** solenoidal vector fields can be expressed as the curl of some other vector field. Note this means that:

The curl of **any** vector field **always** results in a solenoidal field!

Note if we **combine** these two previous equations, we get a **vector identity**:

$$\nabla \cdot \nabla x \mathbf{A}(\bar{r}) = 0$$

a result that is always true for **any** and **every** vector field $\mathbf{A}(\bar{r})$.

Note this result is **analogous** to the identify derived from conservative fields:

$$\nabla x \nabla g(\bar{r}) = 0$$

for **all** scalar fields $g(\bar{r})$.

3. Now, let's recall the **divergence theorem**:

$$\iiint_V \nabla \cdot \mathbf{A}(\bar{r})\, dv = \oiint_S \mathbf{A}(\bar{r}) \cdot \overline{ds}$$

If the vector field $\mathbf{A}(\bar{r})$ is **solenoidal**, we can write this theorem as:

$$\iiint_V \nabla \cdot \mathbf{S}(\bar{r})\, dv = \oiint_S \mathbf{S}(\bar{r}) \cdot \overline{ds}$$

But of course, the divergence of a solenoidal field is **zero** ($\nabla \cdot \mathbf{S}(\bar{r}) = 0$)!

As a result, the **left** side of the divergence theorem is zero, and we can conclude that:

$$\oiint_S \mathbf{S}(\bar{r}) \cdot \overline{ds} = 0$$

In other words the **surface** integral of **any** and **every** solenoidal vector field across a **closed** surface is equal to zero.

Note this result is **analogous** to evaluating a line integral of a conservative field over a closed contour

$$\oint_C \mathbf{C}(\bar{r}) \cdot \overline{d\ell} = 0$$

Lets **summarize** what we know about **solenoidal** vector fields:

1. **Every** solenoidal field can be expressed as the **curl** of some **other** vector field.

2. The curl of **any** and **all** vector fields always results in a solenoidal vector field.

3. The **surface integral** of a solenoidal field across any **closed** surface is equal to **zero.**

4. The **divergence** of every solenoidal vector field is equal to **zero**.

5. The divergence of a vector field is zero **only** if it is **solenoidal**.

The Laplacian

Another differential operator used in electromagnetics is the **Laplacian** operator. There is both a **scalar** Laplacian operator, and a **vector** Laplacian operator. Both operations, however, are expressed in terms of derivative operations that we have **already** studied !

The Scalar Laplacian

The scalar Laplacian is simply the **divergence** of the **gradient** of a scalar field:

$$\nabla \cdot \nabla g(\bar{\mathbf{r}})$$

The scalar Laplacian therefore both **operates** on a scalar field and **results** in a scalar field.

Often, the Laplacian is denoted as "∇^2", i.e.:

$$\nabla^2 g(\bar{\mathbf{r}}) \doteq \nabla \cdot \nabla g(\bar{\mathbf{r}})$$

From the expressions of divergence and gradient, we find that the scalar Laplacian is expressed in **Cartesian** coordinates as:

$$\nabla^2 g(\bar{\mathbf{r}}) = \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial x^2} + \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial y^2} + \frac{\partial^2 g(\bar{\mathbf{r}})}{\partial z^2}$$

The scalar Laplacian can likewise be expressed in **cylindrical** and **spherical** coordinates; results given on **page 53** of your book.

The Vector Laplacian

The vector Laplacian, denoted as $\nabla^2 \mathbf{A}(\bar{r})$, both **operates** on a vector field and **results** in a vector field, and is defined as:

$$\nabla^2 \mathbf{A}(\bar{r}) \doteq \nabla(\nabla \cdot \mathbf{A}(\bar{r})) - \nabla \times \nabla \times \mathbf{A}(\bar{r})$$

Q: *Yikes! Why the heck is this* ***mess*** *referred to as the* ***Laplacian*** *?!?*

A: If we evaluate the above expression for a vector expressed in the **Cartesian** coordinate system, we find that the vector Laplacian is:

$$\nabla^2 \mathbf{A}(\bar{r}) = \nabla^2 A_x(\bar{r})\hat{a}_x + \nabla^2 A_y(\bar{r})\hat{a}_y + \nabla^2 A_z(\bar{r})\hat{a}_z$$

In other words, we evaluate the vector Laplacian by evaluating the **scalar** Laplacian of each Cartesian **scalar** component!

However, expressing the vector Laplacian in the **cylindrical** or **spherical** coordinate systems is **not** so straightforward—use instead the **definition** shown above!

Helmholtz's Theorems

Consider a **differential equation** of the following form:

$$g(t) = \frac{d\,f(t)}{dt}$$

where $g(t)$ is an **explicit** known function, and $f(t)$ is the **unknown** function that we seek.

For example, the differential equation :

$$3t^2 + t - 1 = \frac{d\,f(t)}{dt}$$

has a **solution**:

$$f(t) = t^3 + \frac{t^2}{2} - t + c$$

Thus, the **derivative** of $f(t)$ provides sufficient knowledge to determine the original function $f(t)$ (to within a constant).

An interesting question, therefore, is whether knowledge of the **divergence** and or **curl** of a vector field is **sufficient** to determine the original vector field.

For example, say we **don't** know the expression for vector field $\mathbf{A}(\bar{r})$, but we **do** know its divergence is some scalar function $g(\bar{r})$:

$$\nabla \cdot \mathbf{A}(\bar{r}) = g(\bar{r})$$

Can we, then, **determine** the vector field $\mathbf{A}(\bar{r})$? For example, can $\mathbf{A}(\bar{r})$ be determined from the expression:

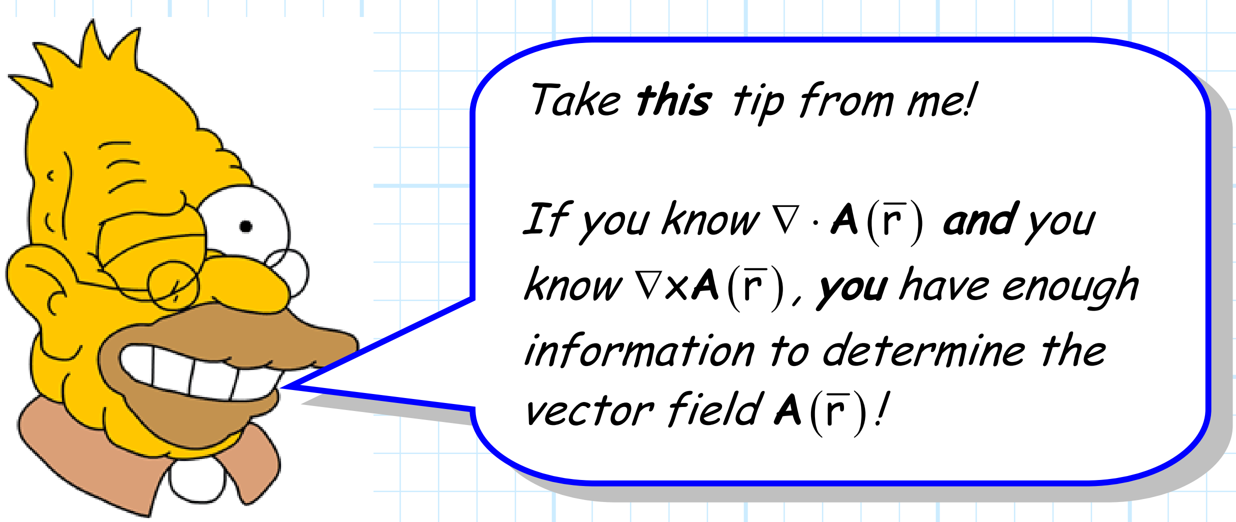
$$\nabla \cdot \mathbf{A}(\bar{r}) = x(y^2 - z^3) \quad ??$$

On the other hand, perhaps the knowledge of the **curl** is sufficient to find $\mathbf{A}(\bar{r})$, i.e.:

$$\nabla \times \mathbf{A}(\bar{r}) = \cos\frac{z\pi}{y}\,\hat{a}_x + (x^2 - 6)\,\hat{a}_y + e^{-\left(x/y\right)}\,\hat{a}_z$$

therefore $\mathbf{A}(\bar{r})$=????

It turns out that **neither** the knowledge of the divergence **nor** the knowledge of the curl **alone** is sufficient to determine a vector field. However, knowledge of **both** the curl and divergence of a vector field is sufficient!



Q: But **why** do we need knowledge of **both** the divergence and curl of a vector field in order to determine the vector field?

A: I know the answer to that as well!

*Its because **every** vector field can be written as the **sum** of a **conservative** field and a **solenoidal** field!*

That's correct! **Any** and **every** possible vector field $\mathbf{A}(\bar{r})$ can be expressed as the sum of a **conservative** field ($\mathbf{C}_A(\bar{r})$) and a **solenoidal** field ($\mathbf{S}_A(\bar{r})$) :

$$\mathbf{A}(\bar{r}) = \mathbf{C}_A(\bar{r}) + \mathbf{S}_A(\bar{r})$$

Note then if $\mathbf{C}_A(\bar{r}) = 0$, the vector field $\mathbf{A}(\bar{r}) = \mathbf{S}_A(\bar{r})$ is **solenoidal**. Likewise, if $\mathbf{S}_A(\bar{r}) = 0$ the vector field $\mathbf{A}(\bar{r}) = \mathbf{C}_A(\bar{r})$ is **conservative**.

Of course, if **neither** term is zero (i.e., $\mathbf{C}_A(\bar{r}) \neq 0$ and $\mathbf{S}_A(\bar{r}) \neq 0$), the vector field $\mathbf{A}(\bar{r})$ is **neither** conservative **nor** solenoidal!

Consider then what happens when we take the **divergence** of a vector field $\mathbf{A}(\bar{r})$:

$$\begin{aligned}\nabla \cdot \mathbf{A}(\bar{r}) &= \nabla \cdot \mathbf{C}_A(\bar{r}) + \nabla \cdot \mathbf{S}_A(\bar{r}) \\ &= \nabla \cdot \mathbf{C}_A(\bar{r}) + 0 \\ &= \nabla \cdot \mathbf{C}_A(\bar{r})\end{aligned}$$

Look what happened! Since the divergence of a solenoidal field is **zero**, the divergence of a general vector field $\mathbf{A}(\bar{r})$ **really** just tells us the divergence of its **conservative** component.

→ The divergence of a vector field tells us **nothing** about its solenoidal component $\mathbf{S}_A(\bar{r})$!

Thus, from $\nabla \cdot \mathbf{A}(\bar{r})$ we **can** determine $\mathbf{C}_A(\bar{r})$, but we haven't a **clue** about what $\mathbf{S}_A(\bar{r})$ is!

Likewise, the curl of $\mathbf{A}(\bar{r})$ is:

$$\begin{aligned}\nabla \times \mathbf{A}(\bar{r}) &= \nabla \times \mathbf{C}_A(\bar{r}) + \nabla \times \mathbf{S}_A(\bar{r}) \\ &= 0 + \nabla \times \mathbf{S}_A(\bar{r}) \\ &= \nabla \times \mathbf{S}_A(\bar{r})\end{aligned}$$

Look what happened! Since the **curl** of a conservative field is **zero**, the curl of a general vector field $\mathbf{A}(\bar{r})$ **really** just tells us the curl of its **solenoidal** component.

→ The curl of a vector field tells us **nothing** about its conservative component $\mathbf{C}_A(\bar{r})$!

Thus, from $\nabla x \mathbf{A}(\bar{r})$ we can determine $\mathbf{S}_A(\bar{r})$, but we haven't a **clue** about what $\mathbf{C}_A(\bar{r})$ is!

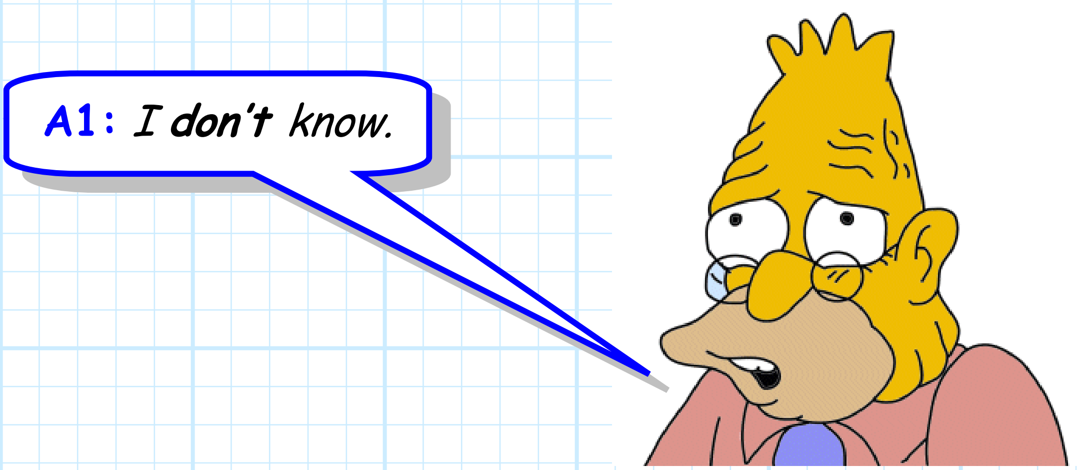
CONCLUSION: We require knowledge of **both** $\nabla \cdot \mathbf{A}(\bar{r})$ (for $\mathbf{C}_A(\bar{r})$) and $\nabla x \mathbf{A}(\bar{r})$ (for $\mathbf{S}_A(\bar{r})$) to determine the vector field $\mathbf{A}(\bar{r})$.

From a **physical** stand point, this makes perfect sense!

Recall that we determined the curl $\nabla x \mathbf{A}(\bar{r})$ identifies the **rotational sources** of vector field $\mathbf{A}(\bar{r})$, while the divergence $\nabla \cdot \mathbf{A}(\bar{r})$ identifies the **divergent** (or convergent) **sources**.

Once we know the **sources** of vector field $\mathbf{A}(\bar{r})$, we can of course **find** vector field $\mathbf{A}(\bar{r})$.

Q: Exactly **how** do we find $\mathbf{A}(\bar{r})$ from its sources ($\nabla \cdot \mathbf{A}(\bar{r})$ and $\nabla x \mathbf{A}(\bar{r})$) ??



A2: Note the **sources** of a vector field are determined from **derivative** operations (i.e., divergence and curl) on the vector field.

We can therefore conclude that a vector field $\mathbf{A}(\bar{r})$ can be determined from its sources with **integral** operations!

We'll learn **much more** about integrating sources later in the course!